ECE 102 Midterm

[redacted]

TOTAL POINTS

97 / 106

QUESTION 1

Signal/System/Convolution 35 pts

1.1 (a)i 5 / 5

✓ **+ 5 pts** Correct

+ 4 pts Correctly computed periods of both signals and its ratio, but with insufficient justification

+ 2 pts Attempted to justify the signal is not periodic, with insufficient or incorrect justification

+ 3.5 pts Stated no common period, without computing or mentioning the ratio of periods

+ 0 pts Incorrect answer or no Justification

1.2 (a)ii 5 / 5

✓ **+ 5 pts** Correct

+ 0 pts Incorrect answer or no/incorrect justification

+ 2 pts Insufficient Reasoning to justify the answer

1.3 (a)iii 5 / 5

✓ **+ 5 pts** Correct answer

+ 3 pts One side is simplified correctly

+ 1.5 pts Correctly used the sifting property on the right hand side

+ 1.5 pts Correctly used the sampling property on the left hand side

+ 0 pts Incorrect answer or reasoning

1.4 (b) 10 / 10

✓ **+ 10 pts** Correct

+ 5 pts Correctly identify the linearity property

+ 0 pts Incorrect

1.5 (c) 10 / 10

✓ **+ 10 pts** Correct h(t)

+ 8 pts Correct h(t) for t<2

+ 8 pts Correct shape of h(t) but incorrect amplitude

+ 2 pts correct h(0)

+ 2 pts correct h(1/4)

+ 2 pts correct h(5/8)

+ 2 pts Correct h(t) for t>2

+ 0 pts Incorrect

QUESTION 2

LTI Systems 20 pts

2.1 (a) 4 / 4

✓ **- 0 pts** Correct

- 1 pts minor mistakes

- 2 pts half correct

- 4 pts incorrect

2.2 (b) 10 / 10

✓ **+ 3 pts** use LTI properties.

✓ **+ 3 pts** correct convolution equation/multiplication in frequency domain.

✓ **+ 2 pts** correct convolution results/inverse FT results.

✓ **+ 2 pts** correct time shift for the last step.

+ 0 pts incorrect

2.3 (c) 6 / 6

✓ **- 0 pts** Correct.

- 1 pts constant term incorrect.

- 2 pts time shift incorrect.

- 2 pts inverse FT incorrect.

- 3 pts missing/incorrect exponential/unit step function.

- 4 pts calculate in frequency domain with wrong FT.

- 5 pts incorrect calculation with a correct convolution equation.

- 6 pts incorrect.

QUESTION 3

Fourier Series 20 pts

3.1 (a) 10 / 10

✓ **- 0 pts** **Correct**

- 0.5 pts Did not explicitly use the angular frequency ωg = aωo, or Tg = To/a in the Fourier Series.

- 5 pts Partially correct

- 7.5 pts Incorrect

- 10 pts See comment

- 10 pts No substantive answer

3.2 (b) 7 / 10

- 0 pts Correct

✓ **- 1 pts** **Did not mention divisibility of k by m1 and m2**

- 6.5 pts Incorrect

- 10 pts See comment

- 2 pts Missing alphas or different scale factor (that does not contain t, since it shouldn't)

✓ **- 2 pts** **k or mk instead of k/m**

- 10 pts No substantive answer

QUESTION 4

Fourier Transform 25 pts

4.1 (a) 10 / 10

✓ **+ 10 pts** **Correct**

+ 5 pts Attempted to use inverse FT formula and did many correct algebraic steps.

+ 3 pts Wrote equation of Fourier Transform but did not substitute omega = 0

+ 2 pts Attempted integral incorrectly, or other attempted math with Fourier transform (either of rect/sincs, or inverse FT).

+ 2 pts Explanation of property with no proof, or applying it to sinc(2t) incorrectly.

+ 0 pts Incorrect or no answer

4.2 (b) 5 / 5

✓ **+ 5 pts** **Correct, with Fourier Transform taken correctly.**

+ 4 pts Took the Fourier Transform correctly, did not calculate an area or did so incorrectly.

+ 2 pts Did not compute Fourier transform of sinc(2t) correctly, or other incorrect algebra.

+ 0 pts No appropriate work for partial credit or answer.

4.3 (c) 5 / 5

✓ **+ 5 pts** **Correct, omega_0 > 2*pi, or correct based on their answer to part (b).**

+ 4.5 pts Mistake on the amount of shift (e.g., 4pi instead of 2pi; or did not plug in omega = 0; or did not specify the shift precisely).

+ 3.5 pts Recognize the FT is time-shifted, did not correctly deduce when rect is 0 or other incorrect algebra.

+ 3 pts Recognized the FT is time-shifted.

+ 2 pts Incorrect answer due to incorrect rationale, or work appropriate for partial credit (such as showing some conditions where it's zero).

+ 0 pts No appropriate work for partial credit, or no answer.

4.4 (d) 5 / 5

✓ **+ 5 pts** **Correct, alpha=-1/2, or correct based on their answer to parts b or c. We required you to simplify to a number since we asked for a value -- it wasn't sufficient to keep things in terms of rects and sincs.**

+ 4 pts Would have had the correct answer if recalled sinc(0) = 1 or other minor algebraic constant.

+ 3 pts Didn't simplify rect(0) or sinc(0), but had the correct (or reasonable) equation; or other related error.

+ 2 pts Incorrect answer due to not treating the rect <-> sinc term correctly, or other partial work.

+ 1 pts Attempt to simplify the integral with incorrect arguments; or other incorrect / incomplete arguments.

+ 0 pts No appropriate work for partial credit, or no answer.

QUESTION 5

5 Bonus **0 / 6**

- **- 0 pts** Correct for intended interpretation
- ✓ **- 6 pts No answer or incorrect justification or no justification**
- **- 3 pts** Partially correct
- **- 5 pts** If "any" was interpreted as "some" rather than "every": correct example of some causal system S
- **- 6 pts** See comment

ECE102, Fall 2019
Department of Electrical and Computer Engineering
University of California, Los Angeles

Midterm
Prof. J.C. Kao
TAs: W. Feng, J. Lee & S. Wu

UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 13 Nov 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.
There is an extra blank space on page 16 to show your work if you run out of space on any questions.

Name: [redacted]

Signature: [redacted]

ID#: [redacted]

Problem 1	____ / 35
Problem 2	____ / 20
Problem 3	____ / 20
Problem 4	____ / 25
BONUS	____ / 6 bonus points
Total	____ / 100 points + 6 bonus points

1. **Signal and System Properties + Convolution** (35 points).

(a) (15 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (5 points) $x(t) = \cos\left(\sqrt{3}t\right) + \sin(-3t)$ is a periodic signal. False

Period of $\cos(\sqrt{3}\,t) = \frac{2\pi}{\sqrt{3}}$ $\quad \{ \cos(\sqrt{3}t_1) = \cos(\sqrt{3}t_1 + 2\pi) \}$

Period of $\sin(-3t) = \frac{2\pi}{3}$ $\quad \{ \sin(-3t_1) = \sin(-3t_1 - 2\pi) \}$

There is no rational k_1, k_2 such that

$$k_1 \frac{2\pi}{\sqrt{3}} = k_2 \frac{2\pi}{3}$$

$\therefore$ $x(t)$ is not periodic

ii. (5 points) A signal can be neither energy signal nor power signal. True

Eg $x(t) = t$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$= \int_{-\infty}^{\infty} t^2\, dt = \infty$$

Non-finite energy & non-finite power

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} t^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{T^3}{3} - \left(\frac{-T^3}{3}\right)\right] = \lim_{T\to\infty} \frac{T^2}{3} = \infty$$

iii. (5 points) Let $f(t) * g(t)$ denote the convolution of two signals, $f(t)$ and $g(t)$. Then,

$$f(t)[\delta(t) * g(t)] = [f(t)\delta(t)] * g(t)$$ False

$$f(t)\,[\delta(t) * g(t)] = f(t)\,g(t)$$

$$[f(t)\,\delta(t)] * g(t) = f(0)\cdot\delta(t) * g(t)$$
$$= f(0)\,g(t)$$

$f(t)\,g(t) \neq f(0)\,g(t)$ for a general $f(t)$ & $g(t)$.

(b) (10 points) Determine if the following system is an LTI system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2) \tag{1}$$

Delayed input

$$\text{Output} = \frac{x(t-\tau-1)}{t} + x(t-\tau-2)$$

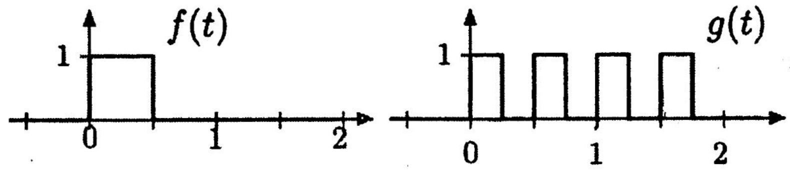
Delayed output

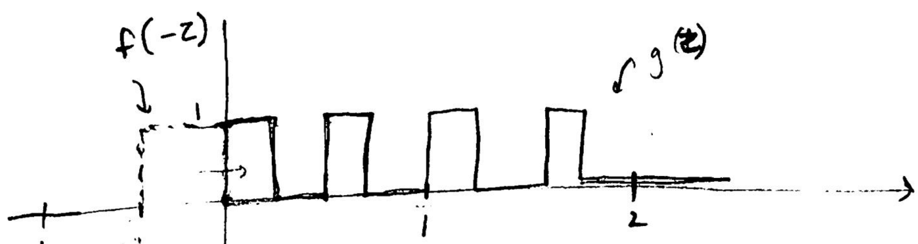
$$y(t-\tau) = \frac{x(t-\tau-1)}{t-\tau} + x(t-\tau-2)$$

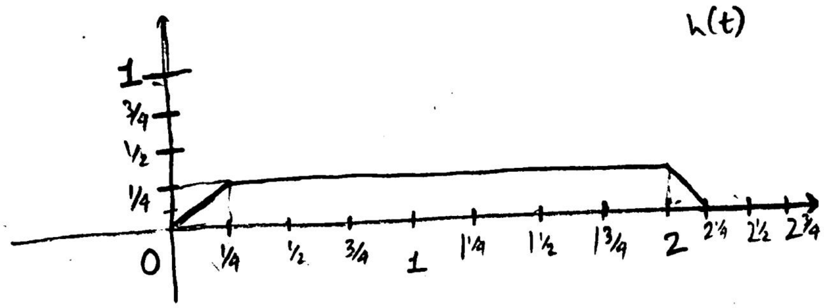
These are not equal, so the system is not time invariant.

$\therefore$ It is not LTI.

(c) (10 points) For signals $f(t)$ and $g(t)$ plotted below, graphically compute the convolution signal $h(t) = f(t) * g(t)$. To receive partial credit, you may show $h(0)$, $h(1/4)$ and $h(5/8)$ in the graph when illustrating the convolution using the "flip and drag" technique.

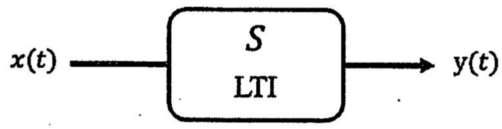






2. **LTI Systems** (20 points).

Consider the following LTI system S:



Consider an input signal $x_1(t) = e^{-2t}u(t-2)$. It is given that

$$x_1(t) \xrightarrow{S} y_1(t)$$

$$\frac{dx_1(t)}{dt} \xrightarrow{S} -2y_1(t) + e^{-2t}u(t)$$

(a) (4 points) Show that:

$$\frac{dx_1(t)}{dt} = -2x_1(t) + e^{-2t}\delta(t-2)$$

$$\frac{d}{dt}\left[e^{-2t}\,u(t-2)\right]$$

$$= -2e^{-2t}\,u(t-2) + e^{-2t}\,\delta(t-2) \quad \{\text{chain rule}\}$$

$$= -2\,x_1(t) + e^{-2t}\,\delta(t-2)$$

(b) (10 points) Find the impulse response $h(t)$ of S.

Hint: Since we have not provided S, we cannot straightforwardly input an impulse into the system and measure the output. One approach is to solve for $h(t)$ by writing the output of S in terms of a convolution when the input is $\mathrm{d}x_1(t)/\mathrm{d}t$, i.e.,

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} * h(t)$$

We know,

$$S\left\{\frac{dx_1(t)}{dt}\right\} = -2y_1(t) + e^{-2t}u(t)$$

$$\Rightarrow \frac{dx_1(t)}{dt} * h(t) = -2y_1(t) + e^{-2t}u(t)$$

$$\Rightarrow \left[-2x_1(t) + e^{-2t}\delta(t-2)\right] * h(t) = -2y_1(t) + e^{-2t}u(t)$$

From properties of convolution,

$$-2\, x_1(t) * h(t) + \left(e^{-2t}\delta(t-2)\right) * h(t) = -2y_1(t) + e^{-2t}u(t)$$

Now, $x_1(t) * h(t) = y_1(t)$ {given}

$$\Rightarrow \left[e^{-2t}\delta(t-2)\right] * h(t) = e^{-2t}u(t)$$

$$\Rightarrow e^{-4}\delta(t-2) * h(t) = e^{-2t}u(t)$$

$$\Rightarrow h(t-2) = e^{-2(t-2)}u(t)$$

$$\therefore h(t) = e^{-2t}u(t+2)$$

(c) (6 points) Consider a new system, S_2, whose impulse response is $h_2(t) = e^{-3t}u(t+3)$. Find this system's output to the following input signal:

$$x_2(t) = \cos\left(\frac{\pi}{4}t\right)\delta(t-1)$$

$$x_2(t) = \cos\left(\frac{\pi}{4}t\right)\delta(t-1)$$

$$= \cos\left(\frac{\pi}{4}\right)\delta(t-1) = k\,\delta(t-1) \quad \left\{ \begin{array}{l} \text{where} \\ k = \cos(\pi/4) \end{array} \right\}$$

$$y_2(t) = x_2(t) * h_2(t)$$

$$= k\,\delta(t-1) * e^{-3t}u(t+3)$$

$$= k\int_{-\infty}^{\infty} e^{-3\tau}u(\tau+3)\,\delta(t-\tau-1)\,d\tau$$

$$= k\int_{-3}^{\infty} e^{-3\tau}\,\delta(t-1-\tau)\,d\tau$$

$$= k\,e^{-3(t-1)}\int_{-3}^{\infty}\delta(t-1-\tau)\,d\tau$$

$$\Rightarrow \quad y_2(t) = \begin{cases} \cos(\pi/4)\,e^{-3(t-1)}, & t > -2 \\ 0, & t < -2 \end{cases}$$

$$= \cos\left(\pi/4\right)e^{-3(t-1)}\,u(t+2)$$

3. **Fourier Series** (20 points).

(a) (10 points) Let the Fourier Series coefficients of $f(t)$ be denoted f_k, and the Fourier Series coefficients of $g(t)$ denoted g_k. Let T_o be the period of $f(t)$. If $g(t) = f(a(t-b))$, where $a > 0$, show that

$$g_k = e^{-j2\pi \frac{ab}{T_o} k} f_k.$$

Period of $g(t) = \frac{T_0}{a} = T_1$

$$f_k = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} f(t) \, e^{-jk\omega_0 t} \, dt \longrightarrow (1)$$

$$g_k = \frac{1}{T_1} \int_0^{T_1} g(t) \, e^{-jk\omega_1 t} \, dt$$

$$= \frac{a}{T_0} \int_0^{T_0/a} f(a(t-b)) \, e^{-jk \frac{2\pi a t}{T_0}} \, dt$$

Let $a(t-b) = t'$

$\Rightarrow a\,dt = dt'$

$$\Rightarrow g_k = \frac{a}{T_0} \int_{-ab}^{-ab+T_0} f(t') \, e^{-jk \frac{2\pi}{T_0}(t'+ab)} \, \frac{dt'}{a}$$

$$= e^{-j 2\pi \frac{ab}{T_0} k} \underbrace{\left[\frac{1}{T_0} \int_{-ab}^{-ab+T_0} f(t') \, e^{-jk\frac{2\pi}{T_0} t'} \, dt' \right]}_{\text{from (1), at } \tau = -ab, \text{ this is } f_k}$$

$$\boxed{\Rightarrow g_k = e^{-j 2\pi \frac{ab}{T_0} k} f_k}$$

(b) (10 points) Let the Fourier Series coefficients of $x(t)$ and $y(t)$ be x_k and y_k respectively, with respective periods T_1 and T_2. We define $f(t) = \alpha_1 x(t) + \alpha_2 y(t)$ with non-zero α_1, α_2, with period $T_o = m_1 T_1 = m_2 T_2$. What are the Fourier Series coefficients f_k in terms of x_k and y_k?

$$x_k = \frac{1}{T_1} \int_0^{T_2} x(t)\, e^{-jk\frac{2\pi}{T_1}t}\, dt$$

$$y_k = \frac{1}{T_2} \int_0^{T_2} y(t)\, e^{-jk\frac{2\pi}{T_2}t}\, dt$$

$$f_k = \frac{1}{T_o} \int_0^{T_o} f(t)\, e^{-jk\frac{2\pi t}{T_o}}\, dt$$

$$= \frac{1}{T_o} \int_0^{T_o} \alpha_1 x(t) e^{-jk\frac{2\pi t}{T_o}}\, dt + \frac{1}{T_o}\int_0^{T_o} \alpha_2 y(t) e^{-jk\frac{2\pi t}{T_o}}\, dt$$

$$= \underbrace{\frac{\alpha_1}{m_1} \frac{1}{T_1} \int_0^{m_1 T_1} x(t)\, e^{-jk\frac{2\pi}{T_1}\frac{t}{m_1}}\, dt}_{t = m_1 \tau} + \underbrace{\frac{\alpha_2}{m_2} \frac{1}{T_2} \int_0^{m_2 T_2} y(t)\, e^{-jk\frac{2\pi}{T_2}\frac{t}{m_2}}\, dt}_{t = m_2 \tau'}$$

$$= \alpha_1 \frac{1}{T_1} \int_0^{T_1} x(m_1 \tau)\, e^{-jk\frac{2\pi}{T_1}\tau}\, d\tau + \alpha_2 \frac{1}{T_2} \int_0^{T_2} y(m_2 \tau') e^{-jk\frac{2\pi \tau'}{T_2}}\, d\tau'$$

$$\Rightarrow f_k = \alpha_1 x_k + \alpha_2 y_k$$

$\{\because C_k$ for $f(ct) = C_k$ for $f(t)\}$

4. **Fourier Transform** (25 points).
Consider the signal

$$x(t) = \text{sinc}(2t)$$

and let the Fourier transform of $x(t)$ be denoted $X(j\omega)$. We are interested in calculating the area under the curve of $x(t)$.

(a) (10 points) Prove that the following relationship holds.

$$\int_{-\infty}^{\infty} x(t)dt = X(j\omega)|_{\omega=0}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{(by definition)}$$

$$X(j\omega)|_{\omega=0} = \int_{-\infty}^{\infty} x(t) e^{-j0t} dt$$

$$= \int_{-\infty}^{\infty} x(t) dt$$

Hence proved.

(b) (5 points) Use the result of part (a) to calculate:

$$\int_{-\infty}^{\infty} x(t)dt$$

for $x(t) = \text{sinc}(2t)$.

$$\int_{-\infty}^{\infty} x(t)\, dt = X(j\omega)\Big|_{\omega=0}$$

For $x(t) = \text{sinc}(2t)$,

$$\text{sinc}(t) \longleftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\text{sinc}(2t) \longleftrightarrow \frac{1}{2}\,\text{rect}\left(\frac{\omega}{4\pi}\right) \quad \{\text{from property of FT}\}$$

$$\Rightarrow X(j\omega)\Big|_{\omega=0} = \frac{1}{2}\,\text{rect}(0)$$

$$= \frac{1}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)\, dt = 1/2$$

(c) (5 points) Consider the following system:

$$y(t) = e^{-j\omega_0 t}x(t)$$

Let $x(t) = \text{sinc}(2t)$ and consider only $\omega_0 > 0$. Are there any values of ω_0 for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of ω_0 does this hold for?

Now, $\int_{-\infty}^{\infty} y(t)\, dt = Y(j\omega)\Big|_{\omega=0}$

From the relation, $y(t) = e^{-j\omega_0 t}\, x(t)$

$$Y(j\omega) = X(j(\omega+\omega_0))$$

$$= \frac{1}{2}\,\text{rect}\left(\frac{\omega+\omega_0}{4\pi}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} y(t)\, dt = \frac{1}{2}\,\text{rect}\left(\frac{\omega_0}{4\pi}\right) = 0$$

This is true for $\omega_0 > 2\pi$ & $\omega_0 < -2\pi$

Since, $\omega_0 > 0$ (from question statement),

we get $\boxed{\omega_0 \in (2\pi, \infty)}$

(d) (5 points) Consider the following system:

$$y(t) = x(t) + \alpha \mathrm{rect}(t)$$

Let $x(t) = \mathrm{sinc}(2t)$. Are there any values of α for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of α does this hold for?

For given $y(t)$,

$$Y(j\omega) = X(j\omega) + \alpha \,\mathrm{sinc}\left(\omega/2\pi\right)$$

$$Y(j\omega)\Big|_{\omega=0} = X(0) + \alpha\, \mathrm{sinc}(0)$$

$$= \frac{1}{2} + \alpha$$

$\Rightarrow$ For $\int_{-\infty}^{\infty} y(t)\,dt = 0$, i.e., $Y(j\omega)\Big|_{\omega=0} = 0$

$$\boxed{\alpha = \frac{-1}{2}}$$

Bonus (6 points) Suppose $f(t) = \cos(\omega_o t)$ is an eigenfunction of an LTI system S for any ω_o, and S cannot be defined as $S[x(t)] = ax(t)$ for some constant a. Is the system S causal? Justify your answer.

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{\omega_0 t} + e^{-\omega_0 t}\right]$$

If $\quad g(t) = S[f(t)]$

$$\Rightarrow \quad g(t) = \frac{1}{2} S\left[e^{\omega_0 t}\right] + \frac{1}{2} S\left[e^{-\omega_0 t}\right]$$

$$= \frac{1}{2} H(\omega_0) e^{\omega_0 t} + \frac{1}{2} H(-\omega_0) e^{-\omega_0 t}$$

$$H(\omega_0) = \int_{-\infty}^{\infty} h(t)\, e^{-\omega_0 t}\, dt \quad \& \quad H(-\omega_0) = \int_{-\infty}^{\infty} h(t)\, e^{\omega_0 t}\, dt$$

Also, since it is an eigen function,

$$\Rightarrow \quad g(t) = k\left[\frac{1}{2}\left(e^{\omega_0 t} + e^{-\omega_0 t}\right)\right]$$

$$\Rightarrow \quad H(\omega_0) = H(-\omega_0) = k$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} h(t)\, e^{-\omega_0 t}\, dt = \int_{-\infty}^{\infty} h(t)\, e^{\omega_0 t}\, dt$$

$$\Rightarrow \quad h(t) = \text{constant}$$

This is an extra piece of paper to show your work. If you use this space for a question, for that question, please write "Refer to page 16."