Department of Electrical and Computer Engineering Prof. J.C. Kao
University of California, Los Angeles TAs: W. Feng, J. Lee & S. Wu

UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 13 Nov 2019.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	_____	/ 35
Problem 2	_____	/ 20
Problem 3	_____	/ 20
Problem 4	_____	/ 25
BONUS	_____	/ 6 bonus points
Total	_____	/ 100 points + 6 bonus points

1. **Signal and System Properties + Convolution** (35 points).

   (a) (15 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

      i. (5 points) $x(t) = \cos\left(\sqrt{3}t\right) + \sin(-3t)$ is a periodic signal.

      Solution: False.
      $\cos(\sqrt{3}t)$ has a period $T_1 = \frac{2\pi}{\sqrt{3}}$
      $\sin(-3t)$ has a period $T_2 = \frac{2\pi}{|-3|} = \frac{2\pi}{3}$
      Then the ratio

$$\frac{T_1}{T_2} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

      is not rational. In other words, we couldn't find integers m and n such that $T = mT_1 = nT_2$. Therefore, $x(t)$ is not periodic.

      ii. (5 points) A signal can be neither energy signal nor power signal.

      Solution: True.
      Example 1: $x(t) = e^t u(t)$ has infinite energy and infinite power.
      Example 2: $x(t) = tan(t)$ is a periodic signal so it is not energy signal. It also has infinite power.

      iii. (5 points) Let $f(t) * g(t)$ denote the convolution of two signals, $f(t)$ and $g(t)$. Then,

$$f(t)[\delta(t) * g(t)] = [f(t)\delta(t)] * g(t)$$

      Solution: False. The left hand side:

$$f(t)[\delta(t) * g(t)] = f(t)g(t)$$

      While the right hand side is

$$[f(t)\delta(t)] * g(t) = [f(0)\delta(t)] * g(t) = f(0)g(t)$$

(b) (10 points) Determine if the following system is an LTI system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2) \tag{1}$$

Solution:
Linearity: Suppose we have two input signals $x_1(t)$, $x_2(t)$ and output signals $y_1(t)$, $y_2(t)$ respectively. If we consider an input signal $x_3(t) = ax_1(t) + bx_2(t)$, then we have the corresponding output signal:

$$\begin{aligned} y_3(t) &= \frac{ax_1(t-1) + bx_2(t-1)}{t} + ax_1(t-2) + bx_2(t-2) \\ &= \left(\frac{ax_1(t-1)}{t} + ax_1(t-2)\right) + \left(\frac{bx_2(t-1)}{t} + bx_2(t-2)\right) \\ &= a\left(\frac{x_1(t-1)}{t} + x_1(t-2)\right) + b\left(\frac{x_2(t-1)}{t} + x_2(t-2)\right) \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Hence, the system is linear.
Time-invariant: Suppose we delay the input signal by t_0, i.e. $x_{t_0}(t) = x(t-t_0)$, the output is:

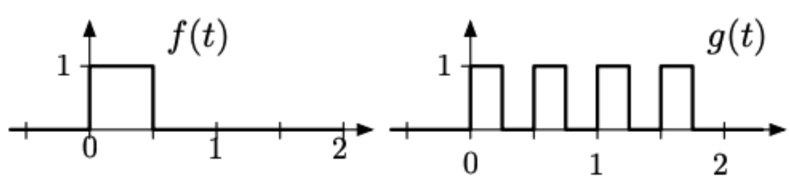
$$y_{t_0}(t) = \frac{x(t-1-t_0)}{t} + x(t-2-t_0)$$

If we delay the output signal by same amount t_0, we have:

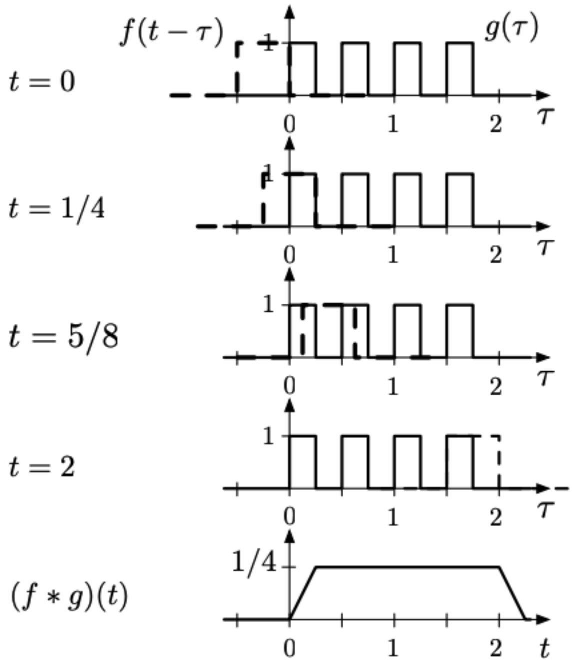
$$y(t-t_0) = \frac{x(t-1-t_0)}{t-t_0} + x(t-2-t_0)$$

We can find that $y_{t_0}(t) \neq y(t-t_0)$. Hence, the system is not time-invariant.

(c) (10 points) For signals $f(t)$ and $g(t)$ plotted below, graphically compute the convolution signal $h(t) = f(t) * g(t)$. To receive partial credit, you may show $h(0)$, $h(1/4)$ and $h(5/8)$ in the graph when illustrating the convolution using the "flip and drag" technique.

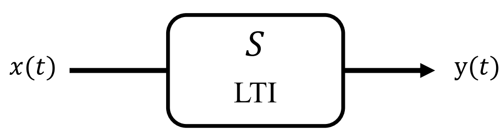


Solution: The graphical convolution using "flip and drag" is illustrated below.
We first flip $f(t)$, to get $f(t-\tau)$, which doesn't overlap with $g(\tau)$ until $t=0$. From $t=0$ to $t=1/4$, the overlapped area increases linearly. As $f(t-\tau)$ shifts further right, it always overlaps the equivalent of one full lobe of $g(\tau)$. The overlapped area keeps constant at $1/4$ until $t=2$, when the area starts to decrease linearly to zero, at $t=2.5$



2. **LTI Systems** (20 points).

Consider the following LTI system S:



Consider an input signal $x_1(t) = e^{-2t}u(t-2)$. It is given that

$$x_1(t) \xrightarrow{\ \ S\ \ } y_1(t)$$
$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \xrightarrow{\ \ S\ \ } -2y_1(t) + e^{-2t}u(t)$$

(a) (4 points) Show that:

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -2x_1(t) + e^{-2t}\delta(t-2)$$

Solution: This is differentiating the input.

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -2e^{-2t}u(t-2) + e^{-2t}\delta(t-2)\\ &= -2x_1(t) + e^{-2t}\delta(t-2)\end{aligned}$$

(b) (10 points) Find the impulse response $h(t)$ of S.

Hint: Since we have not provided S, we cannot straightforwardly input an impulse into the system and measure the output. One approach is to solve for $h(t)$ by writing the output of S in terms of a convolution when the input is ${\mathrm{d}x_1(t)}/{\mathrm{d}t}$, i.e.,

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} * h(t)$$

Solution:
Since the system is LTI, we have:

$$-2x_1(t) + e^{-2t}\delta(t-2) \xrightarrow{S} -2y_1(t) + h(t) \star (e^{-2t}\delta(t-2))$$

Given that

$$\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} \xrightarrow{S} -2y_1(t) + e^{-2t}u(t)$$

we have

$$\begin{aligned}-2y_1(t) + h(t) \star (e^{-2t}\delta(t-2)) &= -2y_1(t) + e^{-2t}u(t)\\ h(t) \star (e^{-2t}\delta(t-2)) &= e^{-2t}u(t)\end{aligned}$$

The left hand side can be calculated by the convolution integral:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-2\tau}\delta(\tau-2)h(t-\tau)d\tau &= \int_{-\infty}^{+\infty} e^{-4}\delta(\tau-2)h(t-2)d\tau\\ &= e^{-4}h(t-2)\int_{-\infty}^{+\infty}\delta(\tau-2)d\tau\\ &= e^{-4}h(t-2)\end{aligned}$$

Therefore, we have:

$$\begin{aligned}e^{-4}h(t-2) &= e^{-2t}u(t)\\ h(t-2) &= e^{-2t+4}u(t)\\ h(t) &= e^{-2t}u(t+2)\end{aligned}$$

(c) (6 points) Consider a new system, S, whose impulse response is $h(t) = e^{-3t}u(t+3)$. Find this system's output to the following input signal:

$$x_2(t) = cos\left(\frac{\pi}{4}t\right)\delta(t-1)$$

Solution: Using the sampling property, we can simply $x_2(t)$ as:

$$x_2(t) = cos\left(\frac{\pi}{4}t\right)\delta(t-1) = \frac{\sqrt{2}}{2}\delta(t-1)$$

Then we have

$$y_2(t) = h(t) * x_2(t) = e^{-3t}u(t+3) * \frac{\sqrt{2}}{2}\delta(t-1) = \frac{\sqrt{2}}{2}e^{-3t+3}u(t+2)$$

3. **Fourier Series** (20 points).

(a) (10 points) Let the Fourier Series coefficients of $f(t)$ be denoted f_k, and the Fourier Series coefficients of $g(t)$ denoted g_k. Let T_o be the period of $f(t)$. If $g(t) = f(a(t-b))$, where $a > 0$, show that

$$g_k = e^{-j2\pi \frac{ab}{T_o} k} f_k.$$

Solution: We begin with the Fourier Series of $f(t)$, and the substitute $a(t-b)$: for t

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_o t} \\ g(t) = f(a(t-b)) &= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_o a(t-b)} \\ &= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_o at} e^{-jk\omega_o ab} \end{aligned}$$

Let $\omega_g = a\omega_o$ be the angular frequency of $g(t)$. We also know that $\omega_o = \frac{2\pi}{T_o}$. Then we can rewrite the above expression as:

$$\begin{aligned} g(t) &= \sum_{k=-\infty}^{\infty} f_k e^{-jk\omega_o ab} e^{jk\omega_g t} \\ &= \sum_{k=-\infty}^{\infty} g_k e^{jk\omega_g t}, \end{aligned}$$

where

$$g_k = e^{-j2\pi \frac{ab}{T_o} k} f_k.$$

(b) (10 points) Let the Fourier Series coefficients of $x(t)$ and $y(t)$ be x_k and y_k respectively, with respective periods T_1 and T_2. We define $f(t) = \alpha_1 x(t) + \alpha_2 y(t)$ with non-zero α_1, α_2, with period $T_o = m_1 T_1 = m_2 T_2$. What are the Fourier Series Coefficients f_k in terms of x_k and y_k?

Solution: Since $T_o = m_1 T_1$, $\omega_1 = m_1 \omega_o$. Then:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_1 t} \\ &= \sum_{k=-\infty}^{\infty} x_k e^{jkm_1\omega_0 t} \end{aligned}$$

We may introduce a change of variables $n = km_1$ so that we have:

$$x(t) = \sum_{\substack{n=-\infty, \\ n=km_1}}^{\infty} x_{\frac{n}{m_1}} e^{jn\omega_0 t}$$

Likewise, for y(t), using $l = km_2$, we have:

$$y(t) = \sum_{\substack{l=-\infty, \\ l=km_2}}^{\infty} y_{\frac{l}{m_2}} e^{jl\omega_0 t}$$

Then:

$$\begin{aligned} f(t) &= \alpha_1 x(t) + \alpha_2 y(t) \\ &= \alpha_1 \sum_{\substack{n=-\infty, \\ n=km_1}}^{\infty} x_{\frac{n}{m_1}} e^{jn\omega_0 t} + \alpha_2 \sum_{\substack{l=-\infty, \\ l=km_2}}^{\infty} y_{\frac{l}{m_2}} e^{jl\omega_0 t} \\ &= \sum_{\substack{n=-\infty, \\ n=km_1}}^{\infty} \alpha_1 x_{\frac{n}{m_1}} e^{jn\omega_0 t} + \sum_{\substack{l=-\infty, \\ l=km_2}}^{\infty} \alpha_2 y_{\frac{l}{m_2}} e^{jl\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_o t} \end{aligned}$$

Therefore,

$$f_k = \begin{cases} \alpha_1 x_{\frac{k}{m_1}} + \alpha_2 x_{\frac{k}{m_2}}, & k \text{ a multiple of } m_1 \text{ and } m_2 \\ \alpha_1 x_{\frac{k}{m_1}}, & k \text{ a multiple of } m_1 \text{ but not } m_2 \\ \alpha_2 x_{\frac{k}{m_2}}, & k \text{ a multiple of } m_2 \text{ but not } m_1 \\ 0, & \text{else} \end{cases}$$

4. **Fourier Transform** (25 points).
Consider the signal

$$x(t) = \text{sinc}(2t)$$

and let the Fourier transform of $x(t)$ be denoted $X(j\omega)$. We are interested in calculating the area under the curve of $x(t)$.

(a) (10 points) Prove that the following relationship holds.

$$\int_{-\infty}^{\infty} x(t)dt = X(j\omega)|_{\omega=0}$$

Solution: The Fourier transform of $x(t)$ is:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Therefore, when $\omega = 0$,

$$\begin{aligned} X(j\omega)|_{\omega=0} &= \int_{-\infty}^{\infty} x(t)e^{-j\cdot 0\cdot t}dt \\ &= \int_{-\infty}^{\infty} x(t)dt \end{aligned}$$

(b) (5 points) Use the result of part (a) to calculate:

$$\int_{-\infty}^{\infty} x(t)dt$$

for $x(t) = \text{sinc}(2t)$.

Solution: From our FT table, we have that

$$\text{sinc}(t) \iff \text{rect}(\omega/2\pi)$$

Using the time scaling property,

$$\text{sinc}(2t) \iff \frac{1}{2}\text{rect}(\omega/4\pi)$$

Therefore, the area is equal to 1/2.

(c) (5 points) Consider the following system:

$$y(t) = e^{-j\omega_0 t}x(t)$$

Let $x(t) = \text{sinc}(2t)$ and consider only $\omega_0 > 0$. Are there any values of ω_0 for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of ω_0 does this hold for?

Solution: Multiplication by a complex exponential in the time domain is shifting in the frequency domain by ω_0. Since $\text{sinc}(2t) \iff \frac{1}{2}\text{rect}(\omega/4\pi)$, then $X(j\omega)$ takes on a value of 1/2 between -2π and 2π but is zero everywhere else. The integral of $y(t)$ will be equal to zero when this rect is shifted such that it is zero at $\omega = 0$. This occurs for a shift of 2π or greater. Therefore this integral is zero whenever $\omega_0 > 2\pi$.

(d) (5 points) Consider the following system:

$$y(t) = x(t) + \alpha\text{rect}(t)$$

Let $x(t) = \text{sinc}(2t)$. Are there any values of α for which

$$\int_{-\infty}^{\infty} y(t)dt = 0$$

and if so, what value(s) of α does this hold for?

Solution: The Fourier Transform of $\text{rect}(t)$ is $\text{sinc}(\omega/2\pi)$, which is equal to 1 at $\omega = 0$. From part (b), the Fourier transform of $\text{sinc}(2t)$ is $\frac{1}{2}\text{rect}(\omega/4\pi)$, which is equal to 1/2 at $\omega = 0$. Therefore, if $\alpha = -1/2$, then $Y(j\omega) = 0$ at $\omega = 0$.

Bonus (6 points) Suppose $x(t) = \cos(\omega_o t)$ is an eigenfunction of an LTI system S for any ω_o, and S cannot be defined as $S[x(t)] = ax(t)$ for some constant a. Is the system S causal? Justify your answer.

Solution: We can write $x(t)$ as $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t} = \frac{1}{2}e^{j\omega_o t} + \frac{1}{2}e^{-j\omega_o t}$. Then the output $y = S[x(t)]$ is:

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} H(jk\omega_o) c_k e^{jk\omega_o t} \\ &= \frac{1}{2}H(j\omega_o)e^{j\omega_o t} + \frac{1}{2}H(-j\omega_o)e^{-j\omega_o t} \end{aligned}$$

In order for $y(t) = ax(t)$ to be satisfied, we need $H(j\omega_o) = H(-j\omega_o)$ to be true for all ω_o, and $H(j\omega)$ is even. This also implies that $h(t)$ is even as well. Since $h(t) \neq 0$, $h(t) \neq a\delta(t)$, and $E_h > 0$, then there exists a value of t for which $h(t) \neq 0$ and $h(-t) \neq 0$. Therefore, S is non-causal.