Problem 1 (19 points)

(a) (9 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (3 points) If $x(t)$ is an energy signal, then $y(t) = x(t) + 1$ is also an energy signal.

Since $x(t)$ is energy, $\int_{-\infty}^{\infty} |x(t)|^2 dt$ is finite

$\int_{-\infty}^{\infty} |x(t)+1|^2$ is not necessarily finite

for example: $u(t)e^{-t}$ approaches zero at $t=\infty$ and is finite

$u(t)e^{-t}+1$ approaches one at $t \to \infty$ and is not finite

false

ii. (3 points) If $x(t)$ is an even signal, then $y(t) = x(t-1)$ is also an even signal.

$x_1(t) = \frac{1}{2}(x(t) + x(-t))$

$y(t) = x_1(t-1) = \frac{1}{2}(x(t-1) + x(-(t-1)))$

$= \frac{1}{2}(x(t-1) + x(-t+1))$

not in the proper form for even signal

false

iii. (3 points) If the input to an LTI system is periodic, then its output is also periodic.

Consider integrator: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
which is L.T.I

Consider periodic input $\cos(\omega_0 t)$

$y(t) = \int_{-\infty}^{t} \cos(\omega_0 \tau)\, d\tau$

$= 0$ if $kt = \frac{2\pi}{\omega_0}$

$=$ some constant $c \in (-1, 1)$ if $kt \neq \frac{2\pi}{\omega_0}$

in either case, not periodic

false

(b) (10 points) Is the following system linear? Is it time invariant? (Check both properties). Explain your answer.

$$y(t) = \begin{cases} x(t-1), & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

to check linearity, see if an input violates:

consider $x(0.5) \Rightarrow y = 0$

$x(0.5 + 0.25) = x(0.75) \Rightarrow y = 0$

$x(0.5) + 0.25 = 0 + 0.25 = 0.25$

$0.25 \neq 0$

not linear

consider $x(t-\tau)$, does it imply $y(t-\tau)$

~~output = { x(t-τ-1) if t-τ ≥ 1 ; 0 else }~~

$y(t) = x(t-1)\,u(t-1)$

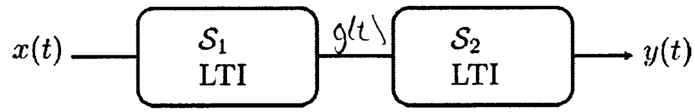
output $= x(t-\tau-1)\,u(t-1)$

$y(t-\tau) = x(t-\tau-1)\,u(t-\tau-1)$

$x(t-\tau-1)u(t-1) \neq x(t-\tau-1)u(t-\tau-1)$

not time invariant

Problem 2 (17 points) Consider the series cascade of the following two systems:



The system S_1 is LTI with impulse response

$$h_1(t) = \int_{-\infty}^{t} u(\tau)\delta(\tau - 2)d\tau$$

The system S_2 is also LTI, with unknown impulse response $h_2(t)$ that we need to find. We are also given that, when the input $x(t)$ is $\delta(t)$, the output $y(t)$ is $r(t-3) + u(t-2)$.

Note: $r(t-3)$ is the ramp function delayed by 3.

This question continues on the next page.

given $\qquad\qquad\qquad\qquad$ given

$$\delta(t) * h_1(t) = g(t) \qquad g(t) * h_2(t) = y(t)$$

$$\delta(t) * h_1(t) = \delta(t) * \int_{-\infty}^{t} u(\tau)\delta(\tau-2)d\tau = u(t-2)$$

$$g(t) = u(t-2)$$

$$u(t-2) * h_2(t) = r(t-3) + u(t-2)$$

$$u(t-2) \Leftrightarrow e^{-j\omega 2}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = e^{-j\omega 2}\pi\delta(\omega) + \frac{e^{-j\omega 2}}{j\omega}$$

$$\frac{d}{dt}r(t-3) = u(t-3) \Leftrightarrow (j\omega)e^{-j\omega 3}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = j\omega\pi e^{-j\omega 3}\delta(\omega) + e^{-j\omega 3}$$

$$\left[e^{-j\omega 2}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right] H_2(\omega) = (j\omega)e^{-j\omega 3}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$
$$= e^{-j\omega 2} e^{-j\omega}(j\omega)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)$$

$$H_2(\omega) = j\omega e^{-j\omega}$$

$$h_2(t) = \frac{d}{dt} x(t-1)$$

(a) (11 points) Find the impulse response $h_2(t)$ of the system S_2 and determine if the system S_2 is causal.

Work on previous page:

$$h_2(t) = \frac{d}{dt} x(t-1)$$

Since $h_2(t) = 0$ for all $t < 0$, and it only contains present and past

the system is causal

(b) (6 points) Find the output $y(t)$ to the following input:

$$x(t) = (1 + e^{-t})\delta(t + 1)$$

$$h_{eq} = h_1(t) * h_2(t) = \left[\int_{-\infty}^{t} u(\tau)\,\delta(\tau-2)\,d\tau\right] * \left[\frac{d}{dt}x(t-1)\right]$$

$$= u(t-2) * \frac{d}{dt}x(t-1)$$

$$= \frac{d}{dt}u(t-2) * x(t-1)$$

$$x(t) = \delta(t+1) + \delta(t+1)e^{-t} = \delta(t+1) + e^{1}$$

$$\delta(t+1) \Leftrightarrow e^{j\omega}$$

$$1 \Leftrightarrow 2\pi\,\delta(\omega), \quad e\cdot 1 \Leftrightarrow 2\pi e\,\delta(\omega)$$

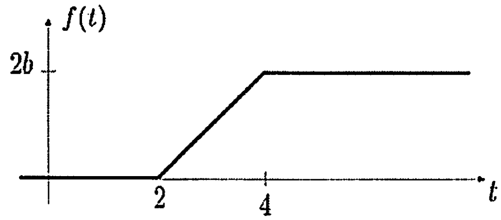
$$X(\omega) = e^{j\omega} + 2\pi e\,\delta(\omega)$$

$$Y(\omega) = X(\omega)\cdot H_1(\omega)\cdot H_2(\omega)$$

$$Y(\omega) = \left[e^{j\omega} + 2\pi e\,\delta(\omega)\right]\left[e^{-j\omega 2}\left(\pi\,\delta(\omega) + \frac{1}{j\omega}\right)\right]\left[\frac{d}{dt}X(t-1)\right]$$

Problem 3 (16 points)

(a) (8 points) Consider the signal $f(t)$ shown below:

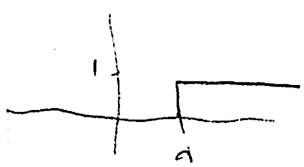


This signal can be written as

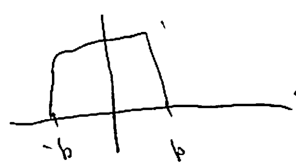
$$u(t-a) * \text{rect}\left(\frac{t}{2b}\right)$$

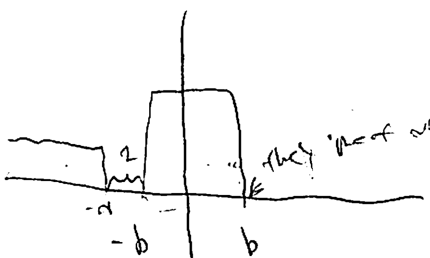
where $a > 0$ and $b > 0$. Find a and b. (Hint: use the flip and drag technique.)

$u(t-a)$



$\text{rect}\left(\frac{t}{2b}\right)$





they meet at $t=4$. length = 2 seconds, $b=1$

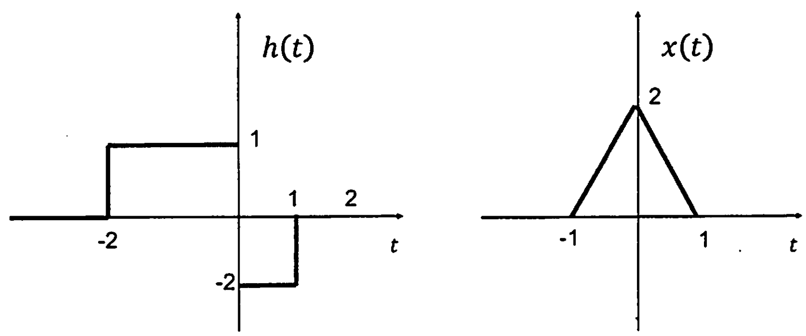
at $t=2$, $a=b$ $\Rightarrow a = b+2$

$a = 1+2 = 3$

$a=3$

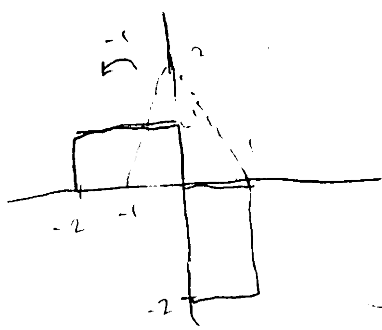
$b=1$

(b) (8 points) An input, $x(t)$, is given to an LTI system with impulse response $h(t)$. Both $x(t)$ and $h(t)$ are shown below.



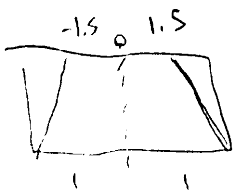
Let $y(t)$ denote the output of the system, i.e., $y(t) = x(t) * h(t)$. Find the value of t at which the output $y(t)$ reaches its maximum value. Determine this maximum value.

Note: to answer this question, you do ***not*** *need to find $y(t)$ for all t.*



maximum area when dragged to the left 1 unit

$t = -1$



area $= 2 \cdot 1 \cdot 1.75 = 3.5$

Problem 4 (20 points)

Consider the following two periodic signals $f(t)$ and $g(t)$. They both have the same period T_0. Let f_k and g_k respectively denote the Fourier series coefficients of $f(t)$ and $g(t)$.

(a) (6 points) If $f(t) = -g\left(t + \frac{T_0}{2}\right)$, how is f_k related to g_k?

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t}$$

$$g(t) = \sum_{k=-\infty}^{\infty} g_k e^{jk\omega_0 t}$$

$$-g(t) = \sum_{k=-\infty}^{\infty} -g_k e^{jk\omega_0 t}$$

$$-g\left(t+\frac{T_0}{2}\right) = \sum_{k=-\infty}^{\infty} -g_k e^{jk\omega_0 \left(t+\frac{T_0}{2}\right)}$$

$$\omega_0 = \frac{2\pi}{T_0} \Rightarrow \frac{2\pi}{T_0}\cdot\frac{T_0}{2} = \pi$$

$$= \sum_{k=-\infty}^{\infty} -g_k e^{jk\omega_0 t} e^{jk\omega_0 \frac{T_0}{2}}$$

$$= \sum_{k=-\infty}^{\infty} -g_k e^{jk\omega_0 t} e^{jk\pi}$$

$$= \sum_{k=-\infty}^{\infty} -g_k e^{jk\omega_0 t} (-1)^k$$

$$= \sum_{k=-\infty}^{\infty} g_k e^{jk\omega_0 t} (-1)^{(k+1)}$$

$$f_k = \begin{cases} g_k & \text{for odd } k \\ -g_k & \text{for even } k \end{cases}$$

(b) (6 points) If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, for what k are the coefficients f_k zero?

$$f(t) = \sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t}$$

$$-f(t) = \sum_{k=-\infty}^{\infty} -f_k e^{jk\omega_0 t}$$

$$-f(t+\tfrac{T_0}{2}) = \sum_{k=-\infty}^{\infty} -f_k e^{jk\omega_0 t} e^{jk\omega_0 \frac{T_0}{2}}$$

$$= -f_k e^{jk\omega_0 t} (-1)^k$$

$$= f_k e^{jk\omega_0 t} (-1)^{k+1}$$

$$e^{jk\omega_0 t} = e^{jk\omega_0 t} (-1)^{k+1}$$

$$1 = (-1)^{k+1} \Rightarrow \boxed{\text{odd } k \text{ values}}$$

(c) (8 points). This question has two parts. *Note: part (c) is independent of parts (a) and (b).*

i. (4 points) Let $f_e(t)$ denote the even part of $f(t)$. Express the Fourier series coefficients of $f_e(t)$ in terms of f_k.

$f_e(t) = \frac{1}{2}(f(t) + f(-t))$

$= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} f_k e^{-jk\omega_0 t}\right)$

$= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} f_k \left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right)\right)$

$= \frac{1}{2}\sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0} (e^{t} + e^{-t})$

let $n = -k$

$= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} f_k e^{jk\omega_0 t} + \sum_{n=-k}^{\infty} f_{-n} e^{jk\omega_0 n}\right)$

$f_{e_k} = \frac{1}{2} f_k + f_{-k}$

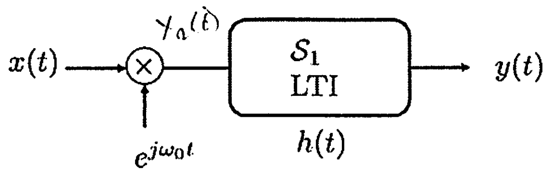
ii. (4 points) Determine the DC component of $f_o(t)$, the odd part of $f(t)$.

Since odd, $x(t) = -x(t)$, therefore its integral will be zero. This corresponds to the D.C. component

0

Problem 5 (28 points)

Consider the following system ($\omega_0 > 0$):



The system S_1 is LTI and $h(t)$ represents its impulse response.

(a) (10 points) Show that the overall system, with input $x(t)$ and output $y(t)$, is not time-invariant.

$x_2(t) = e^{jW_0 t} x(t)$

$y(t) = e^{jW_0 t} x(t) * h(t)$

$Y(W) = e^{jW_0 t} X(W) H(W)$

the factor $e^{jW_0 t}$ results in a of $(W + W_0)$

Assuming arbitrary system $S_A(X(\omega))$

output $= Y(X(W + W_0 + W_1))$

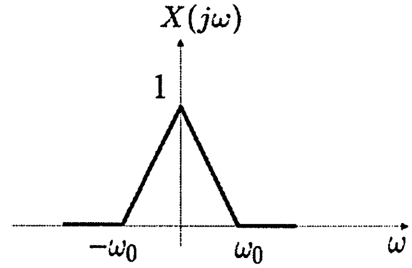
$S_A(X(\omega + W_0) + W_1) = y(x(W + W_0)) + y(W_1) \neq Y(X(W + W_0 + W_1))$

then system is **not time-invariant**

(b) (12 points) Consider the following impulse response for system $\mathcal{S}_1$:

$$h(t) = e^{j\frac{\omega_0}{2}t}\text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$$

We give the system an input $x(t)$, where $x(t)$ has the following Fourier transform $X(j\omega)$:



Find and sketch the Fourier transform $Y(j\omega)$ of the corresponding output $y(t)$. After this, determine (i) if $y(t)$ is real and (ii) if $y(t)$ is even. *Note: you do not need to give an expression for $Y(j\omega)$, a sketch of it is enough. There is space on the next page if needed.*

$Y(\omega) = H(\omega)X(\omega)$

$h(t) = e^{j\frac{\omega_0}{2}t}\,\text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$

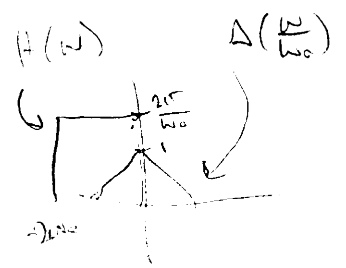
$\text{sinc}(t) \Leftrightarrow \text{rect}\left(\frac{\omega}{2\pi}\right)$

$x(at) \Leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

$\text{sinc}\left(\frac{\omega_0}{2\pi}t\right) \Leftrightarrow \left|\frac{2\pi}{\omega_0}\right| \text{rect}\left(\frac{\omega}{2\pi}\cdot\frac{2\pi}{\omega_0}\right) = \left|\frac{2\pi}{\omega_0}\right| \text{rect}\left(\frac{\omega}{\omega_0}\right)$

$H(\omega) = \left|\frac{2\pi}{\omega_0}\right| \text{rect}\left(\frac{\omega}{\omega_0} + \frac{\omega_0}{2}\right)$

$Y(\omega) = \left|\frac{2\pi}{\omega_0}\right| \text{rect}\left(\frac{\omega}{\omega_0} + \frac{\omega_0}{2}\right)\cdot\Delta\left(\frac{\omega}{\omega_0}\right)$



real

not even

(c) (6 points) Suppose

$$z(t) = y(3t - 2)$$

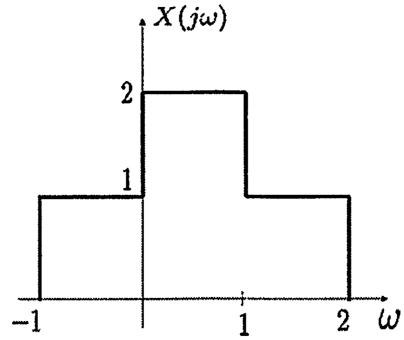
Express $Z(j\omega)$ in terms of $Y(j\omega)$. *Note: part (c) is independent of parts (a) and (b).*

$y(3t-2) = y(3(t-\frac{2}{3}))$

$= \frac{1}{3} Y(\omega) e^{-j\omega \frac{2}{3}}$

BONUS (6 points)

(a) (4 points) The Fourier transform $X(j\omega)$ of a signal $x(t)$ is given as follows:

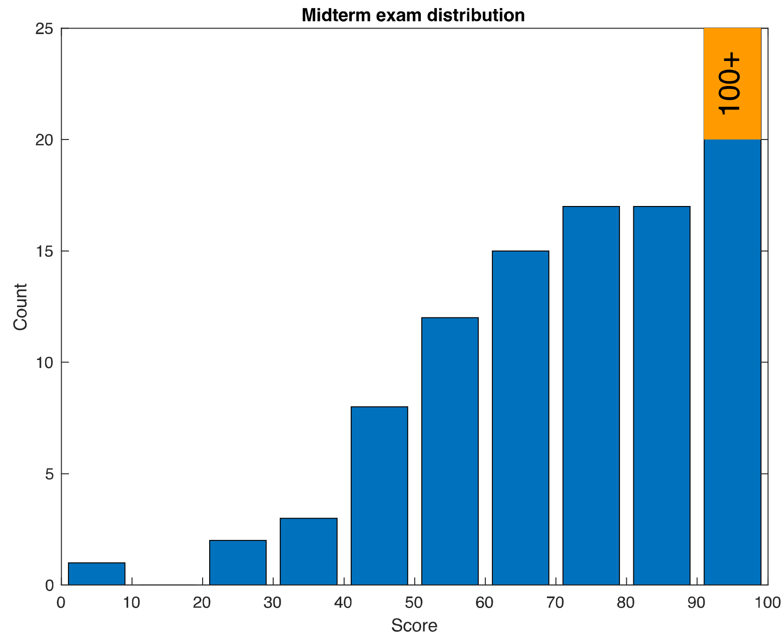


Find the phase of $x^2(t)$.

$\phi = \tan\left(\frac{Im}{Re}\right)$

(b) (2 points) If a signal $x(t)$ is causal with $x(0) = 0$, how can we retrieve $x(t)$ from its even component $x_e(t)$?

ECE 102, Fall 2018 | **MIDTERM STATISTICS**
Department of Electrical and Computer Engineering | Prof. J.C. Kao
University of California, Los Angeles | TAs H. Salami & S. Shahsavari



Statistics: Mean: 72.2 (out of 100), Standard deviation: 20.3, 25th perentile: 56.5, Median: 74.5, 75th percentile: 90, Maximum score: 103.5, Number of exams: 100

Comments:

- If you have grading questions, please submit through Gradescope. Regrades should be submitted if you believe we applied the rubric mistakenly.
- The exam was longer than we intended. We will use this information to calibrate the length and difficulty of the final.
- Given the length of the midterm, I will make the following offer: if you do better on the final exam than the midterm, I will count the final exam score as your midterm score (i.e., your final will count as 60% of your grade; with the HW being 40%).
- Please be reminded that the absolute grading scale may be relaxed. The median final grade in this class will not be lower than a B+.
- Please don't hesitate to contact Prof. Kao if you received a poor score and would like to talk further, or have overall course grade questions.

ECE 102, Fall 2018 **Midterm**

Department of Electrical and Computer Engineering — Prof. J.C. Kao

University of California, Los Angeles — TAs: H. Salami, S. Shahshavari

UCLA True Bruin academic integrity principles apply.
Open: Two pages of cheat sheet allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 14 Nov 2018.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: ____________________

Signature: ____________________

ID#: _________________

Problem 1	_____	/	19
Problem 2	_____	/	17
Problem 3	_____	/	16
Problem 4	_____	/	20
Problem 5	_____	/	28
BONUS	_____	/	6 bonus points
Total	_____	/	100 points + 6 bonus points

Problem 1 (19 points)

(a) (9 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (3 points) If $x(t)$ is an energy signal, then $y(t) = x(t) + 1$ is also an energy signal.

Solution: False
$x(t) + 1$ is not an energy signal, adding a constant to a signal will in general make its energy go to infinity. Consider for instance the finite energy signal: $x(t) = e^{-t}u(t)$, then

$$\int_{-\infty}^{+\infty} |x(t) + 1|^2 dt = \int_{-\infty}^{+\infty} (x^2(t) + 2x(t))dt + \int_{-\infty}^{+\infty} 1dt = \frac{1}{2} + 2 + \infty$$

Therefore, $x(t) + 1$ is not an energy signal.

ii. (3 points) If $x(t)$ is an even signal, then $y(t) = x(t-1)$ is also an even signal.

Solution: False
Consider for instance the unit triangle $x(t) = \Delta(t)$, which is an even function. However, shifting the unit triangle to the right by one will make it defined only over $t \geq 0$, we then obtain $x(t-1) = \Delta(t-1)$, which is not even.

iii. (3 points) If the input to an LTI system is periodic, then its output is also periodic.

Solution: True
If the input is periodic, then it can be written as:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

where c_k's are the Fourier series coefficients of $x(t)$. Then using the eigenfunction property, we obtain the corresponding output:

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha_k c_k e^{j\omega_0 kt}$$

where α_k is the eigenvalue that corresponds to $e^{j\omega_0 kt}$. Therefore, $y(t)$ is also periodic.

(b) (10 points) Is the following system linear? Is it time invariant? (Check both properties). Explain your answer.

$$y(t) = \begin{cases} x(t-1), & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Solutions: We can equivalently write the system as follows:

$$y(t) = x(t-1)u(t-1)$$

Linearity: Suppose to inputs $x_1(t)$ and $x_2(t)$, we respectively get $y_1(t)$ and $y_2(t)$ as outputs. Now if we consider the following input $x_3(t) = ax_1(t) + bx_2(t)$, then its output:

$$y_3(t) = x_3(t-1)u(t-1) = (ax_1(t-1)u(t-1) + bx_2(t-1)u(t-1)) = ay_1(t) + by_2(t)$$

The system is then linear.

Time Invariant:
If we delay the input by τ, i.e. $x_\tau(t) = x(t-\tau)$, the output is then:

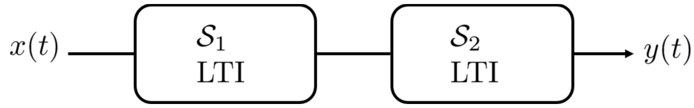
$$y_\tau(t) = x_\tau(t-1)u(t-1) = x(t-1-\tau)u(t-1)$$

Now if we delay the output, we get:

$$y(t-\tau) = x(t-\tau-1)u(t-\tau-1)$$

Since $y(t-\tau) \neq y_\tau(t)$, the system is not time-invariant.

Problem 2 (17 points) Consider the series cascade of the following two systems:



The system $\mathcal{S}_1$ is LTI with impulse response

$$h_1(t) = \int_{-\infty}^{t} u(\tau)\delta(\tau - 2)d\tau$$

The system $\mathcal{S}_2$ is also LTI, with unknown impulse response $h_2(t)$ that we need to find. We are also given that, when the input $x(t)$ is $\delta(t)$, the output $y(t)$ is $r(t-3) + u(t-2)$.

Note: $r(t-3)$ is the ramp function delayed by 3.

This question continues on the next page.

(a) (11 points) Find the impulse response $h_2(t)$ of the system $\mathcal{S}_2$ **and** determine if the system $\mathcal{S}_2$ is causal.

Solution:
We first simplify the impulse response of the first system:

$$h_1(t) = \int_{-\infty}^{t} u(\tau)\delta(\tau-2)d\tau = \int_{-\infty}^{t} u(2)\delta(\tau-2)d\tau = \int_{-\infty}^{t} \delta(\tau-2)d\tau = u(t-2)$$

The impulse response of overall system is given by:

$$h_{eq}(t) = r(t-3) + u(t-2)$$

This is because it is given as the output of the overall system when the input is $\delta(t)$.
When the input is $x(t) = \delta(t)$, the intermediate signal between the two systems is the output of $\mathcal{S}_1$ to the input $\delta(t)$. Therefore, the intermediate signal in this case is: $h_1(t) = u(t-2)$. Therefore, we have the following for system $\mathcal{S}_2$:

Input: $u(t-2) \longrightarrow^{\mathcal{S}_2}$ output: $r(t-3) + u(t-2)$

Since $\mathcal{S}_2$ is LTI, we can deduce its step response (by shifting the output to the left by 2):

Input: $u(t) \longrightarrow^{\mathcal{S}_2}$ output: $r(t-1) + u(t)$

Therefore, the step response of $\mathcal{S}_2$ is:

$$r(t-1) + u(t)$$

Thus, the impulse response of $\mathcal{S}_2$ is:

$$h_2(t) = \frac{d}{dt}(r(t-1) + u(t)) = u(t-1) + \delta(t)$$

Since $h_2(t) = 0$ for $t < 0$, the LTI system $\mathcal{S}_2$ is causal.

Note: We received answers like this: because in $h_2(t)$ we have $t-1$ in $u(t-1)$ and t in $\delta(t)$, the system then depends on past and present values of the input, then it is causal. This is not a right justification, because $h_2(t)$ is not the input-output relationship of the system, we used that justification when we have the mapping from the input to output. However, We can using $h_2(t)$ represent the system in terms of its input-output mapping through convolution:

$$y(t) = h_2(t) * z(t) = \int_{-\infty}^{\infty} h_2(\tau)z(t-\tau)d\tau$$

where $z(t)$ is the input to the second system. Now we check if $y(t)$ depends on past values of input by checking the arguments of z. We have $y(t)$ depends on $z(t-\tau)$ and because $h_2(\tau)$ is zero for $\tau < 0$, τ will always be positive so that $z(t-\tau)$ will always be a past value of input for $y(t)$. This is why we can say that the system is causal.

(b) (6 points) Find the output $y(t)$ to the following input:

$$x(t) = (1 + e^{-t})\delta(t + 1)$$

Solution:
Using the sampling property, we can simplify $x(t)$ as follows:

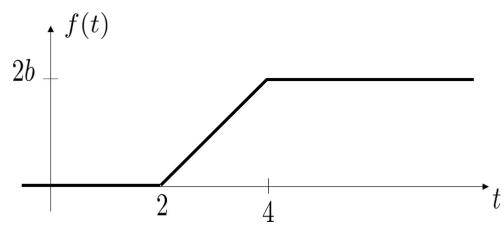
$$x(t) = (1 + e^{-t})\delta(t + 1) = (1 + e)\delta(t + 1)$$

Then,

$$\begin{aligned} y(t) &= h_{eq}(t) * x(t) \\ &= (r(t-3) + u(t-2)) * ((1+e)\delta(t+1)) = (1+e)(r(t-3+1) + u(t-2+1)) \\ &= (1+e)(r(t-2) + u(t-1)) \end{aligned}$$

Problem 3 (16 points)

(a) (8 points) Consider the signal $f(t)$ shown below:

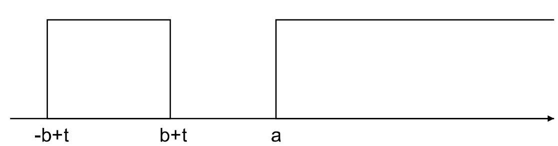


This signal can be written as

$$u(t-a) * \text{rect}\left(\frac{t}{2b}\right)$$

where $a > 0$ and $b > 0$. Find a and b. (Hint: use the flip and drag technique.)

Solution:
Using the flip and drag technique, we have:



$f(t) = 0$ when there is no overlap, i.e., when $b + t < a$ or $t < a - b$.We have $f(t) = 0$ for $t < 2$, therefore

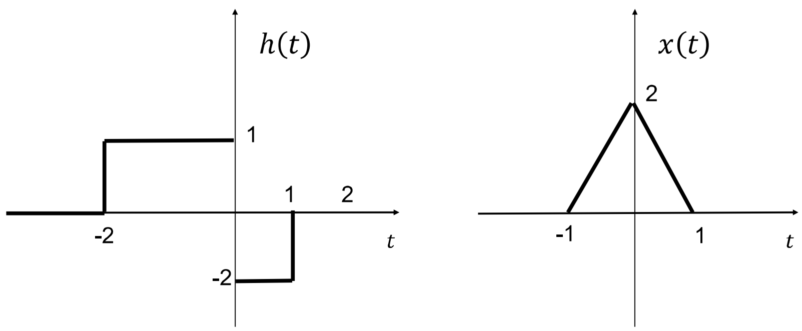
$$a - b = 2$$

The total overlap happens when $-b + t > a$ or $t > a + b$. The function $f(t)$ reaches its maximum at $2b$ and stays at this value for $t > 4$, thus

$$a + b = 4$$

Solving two equation, we get $a = 3$ and $b = 1$.

(b) (8 points) An input, $x(t)$, is given to an LTI system with impulse response $h(t)$. Both $x(t)$ and $h(t)$ are shown below.



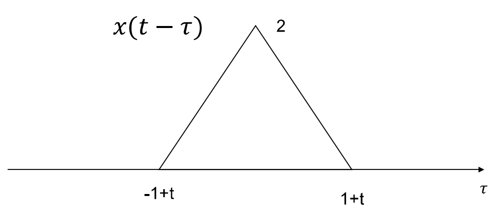
Let $y(t)$ denote the output of the system, i.e., $y(t) = x(t) * h(t)$. Find the value of t at which the output $y(t)$ reaches its maximum value. Determine this maximum value.

*Note: to answer this question, you do **not** need to find $y(t)$ for all t.*

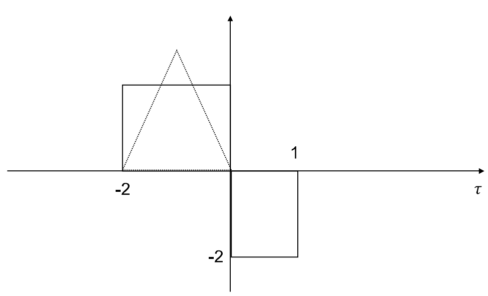
Solution:

We know that

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$



The maximum value of $y(t)$ happens when the triangle totally overlaps with the rectangle part of $h(t)$ that only has positive values, as shown here:



This happens when $1 + t = 0$ therefore, $t = -1$. In this case, the maximum value is the area of the triangle is:

$$y(-1) = \frac{2 \times 2}{2} = 2$$

Problem 4 (20 points)

Consider the following two periodic signals $f(t)$ and $g(t)$. They both have the same period T_0. Let f_k and g_k respectively denote the Fourier series coefficients of $f(t)$ and $g(t)$.

(a) (6 points) If $f(t) = -g\left(t + \frac{T_0}{2}\right)$, how is f_k related to g_k?

Solution: We have:

$$f(t) = \sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 kt} \quad \text{and,} \quad g(t) = \sum_{k=-\infty}^{+\infty} g_k e^{j\omega_0 kt}$$

Now, if $f(t) = -g(t + \frac{T_0}{2})$, then

$$\begin{aligned}
f(t) &= -g(t + \frac{T_0}{2}) \\
&= \sum_{k=-\infty}^{+\infty} -g_k e^{j\omega_0 k(t+\frac{T_0}{2})} \\
&= \sum_{k=-\infty}^{+\infty} -g_k e^{j\omega_0 k\frac{T_0}{2}} e^{j\omega_0 kt} \\
&= \sum_{k=-\infty}^{+\infty} -g_k e^{j\frac{2\pi}{T_0}k\frac{T_0}{2}} e^{j\omega_0 kt} \\
&= \sum_{k=-\infty}^{+\infty} -g_k e^{j\pi k} e^{j\omega_0 kt} = \sum_{k=-\infty}^{+\infty} -g_k (-1)^k e^{j\omega_0 kt} \\
&= \sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 kt}
\end{aligned}$$

Therefore,

$$f_k = -(-1)^k g_k$$

(b) (6 points) If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, for what k are the coefficients f_k zero?

Solution: If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, Then using the previous conclusion, we have:

$$f_k = -(-1)^k f_k$$

Therefore, for even k:

$$f_k = -f_k \rightarrow f_k = 0$$

(c) (8 points) This question has two parts. *Note: part (c) is independent of parts (a) and (b).*

i. (4 points) Let $f_e(t)$ denote the even part of $f(t)$. Express the Fourier series coefficients of $f_e(t)$ in terms of f_k.

Solution:
The even part of the signal is: $f_e(t) = \frac{f(t)+f(-t)}{2}$, $f(-t)$ is also periodic we thus have:

$$\begin{aligned}
f_e(t) &= \frac{f(t)+f(-t)}{2} \\
&= \frac{\sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 kt} + \sum_{k=-\infty}^{+\infty} f_k e^{-j\omega_0 kt}}{2} \\
&= \frac{1}{2}\left(\sum_{k=-\infty}^{+\infty} f_k e^{j\omega_0 kt} + \sum_{k=-\infty}^{+\infty} f_{-k} e^{j\omega_0 kt}\right) \\
&= \frac{1}{2}\sum_{k=-\infty}^{+\infty} (f_k + f_{-k}) e^{j\omega_0 kt} \\
&= \sum_{k=-\infty}^{+\infty} \frac{1}{2}(f_k + f_{-k}) e^{j\omega_0 kt}
\end{aligned}$$

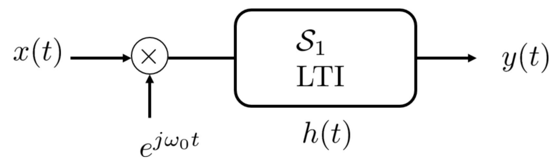
Therefore, the Fourier series coefficients of $f_e(t)$ is $\frac{1}{2}(f_k + f_{-k})$.

ii. (4 points) Determine the DC component of $f_o(t)$, the odd part of $f(t)$.

Solution: The Fourier coefficients of the odd part of the signal is $f_{o,k} = \frac{1}{2}(f_k - f_{-k})$.
Therefore at $k = 0$, $f_{o,0} = \frac{1}{2}(f_0 - f_{-0}) = 0$.
In fact, any odd signal has zero DC component.

Problem 5 (28 points)

Consider the following system ($\omega_0 > 0$):



The system $\mathcal{S}_1$ is LTI and $h(t)$ represents its impulse response.

(a) (10 points) Show that the overall system, with input $x(t)$ and output $y(t)$, is not time-invariant.

Solution: The input-output relationship of the system is given by:

$$y(t) = [e^{j\omega_0 t}x(t)] * h(t) = \int_{-\infty}^{+\infty} h(\tau)e^{j\omega_0(t-\tau)}x(t-\tau)d\tau = e^{j\omega_0 t}\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega_0\tau}x(t-\tau)d\tau$$

If we delay the input, i.e., $x_\alpha(t) = x(t-\alpha)$, the corresponding output is:

$$y_\alpha(t) = e^{j\omega_0 t}\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega_0\tau}x_\alpha(t-\tau)d\tau = e^{j\omega_0 t}\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega_0\tau}x(t-\alpha-\tau)d\tau$$

On the other hand, if we shift the output, we have:

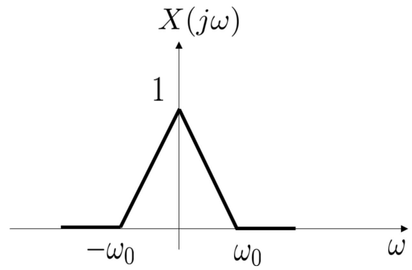
$$y(t-\alpha) = e^{j\omega_0(t-\alpha)}\int_{-\infty}^{+\infty} h(\tau)e^{-j\omega_0\tau}x(t-\alpha-\tau)d\tau$$

Since $y(t-\alpha) \neq y_\alpha(t)$, the system is not TI.

(b) (12 points) Consider the following impulse response for system $\mathcal{S}_1$:

$$h(t) = e^{j\frac{\omega_0}{2}t}\text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$$

We give the system an input $x(t)$, where $x(t)$ has the following Fourier transform $X(j\omega)$:

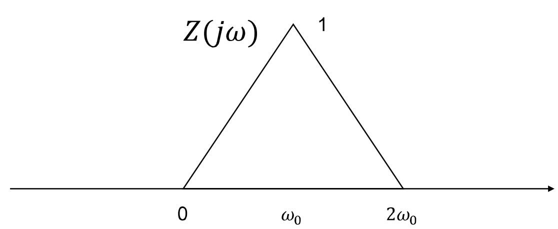


Find and sketch the Fourier transform $Y(j\omega)$ of the corresponding output $y(t)$. After this, determine (i) if $y(t)$ is real and (ii) if $y(t)$ is even. *Note: you do not need to give an expression for $Y(j\omega)$, a sketch of it is enough. There is some space on the next page if needed.*

Solution:
If $z(t) = x(t)e^{j\omega_0 t}$, then using the Fourier transform properties, we have:

$$Z(j\omega) = X(j(\omega - \omega_0))$$

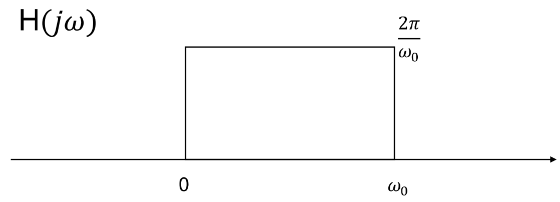
Here is a sketch of $Z(j\omega)$:



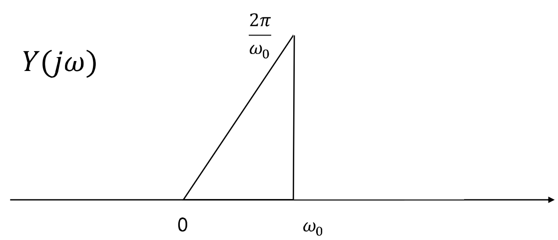
Now using the properties we find the Fourier transform of $h(t)$,

$$\text{sinc}(t) \longleftrightarrow \text{rect}(\frac{\omega}{2\pi})$$
$$\text{sinc}(\frac{\omega_0}{2\pi}t) \longleftrightarrow \frac{2\pi}{\omega_0}\text{rect}(\frac{\omega}{2\pi}.\frac{2\pi}{\omega_0})$$
$$e^{j\frac{\omega_0}{2}t}\text{Sinc}(\frac{\omega_0}{2\pi}t) \longleftrightarrow \frac{2\pi}{\omega_0}\text{rect}(\frac{\omega - \frac{\omega_0}{2}}{\omega_0})$$

Therefore, $H(j\omega)$ is as follows:



Since $y(t) = h(t) * z(t)$, therefore $Y(j\omega) = H(j\omega)X(j\omega)$. Therefore,



Since $Y^*(j\omega) \neq Y(-j\omega)$, $y(t)$ is not real.
Since $Y(j\omega)$ is not even, $y(t)$ is not even.

(c) (6 points) Suppose

$$z(t) = y(3t - 2)$$

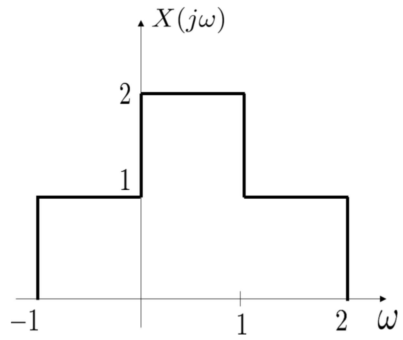
Express $Z(j\omega)$ in terms of $Y(j\omega)$. *Note: part (c) is independent of parts (a) and (b).*

Solution:
Using the properties:

$$\begin{aligned} y(t) &\longleftrightarrow Y(j\omega) \\ y(3t) &\longleftrightarrow \frac{1}{3}Y(\frac{j\omega}{3}) \\ y(3(t - \frac{2}{3})) &\longleftrightarrow \frac{1}{3}e^{-j\frac{2}{3}\omega}Y(\frac{j\omega}{3}) \end{aligned}$$

BONUS (6 points)

(a) (4 points) The Fourier transform $X(j\omega)$ of a signal $x(t)$ is given as follows:



Find the phase of $x^2(t)$.

Solution:
Let $F(j\omega) = X(j(\omega + \frac{1}{2}))$. Thus, $F(j\omega)$ is real and even. Therefore,

$$f(t) = e^{-jt\frac{1}{2}}x(t)$$

is a real function. Thus,

$$x(t) = e^{jt\frac{1}{2}}f(t)$$

and,

$$x^2(t) = e^{jt}f^2(t)$$

Therefore the phase of $x^2(t)$ is t.

(b) (2 points) If a signal $x(t)$ is causal with $x(0) = 0$, how can we retrieve $x(t)$ from its even component $x_e(t)$?

Solution:
Since this signal is causal, therefore:

$$x(t) = 0, \text{ for } t < 0$$

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \begin{cases} \frac{x(-t)}{2}, & t < 0 \\ \frac{x(t)}{2}, & t > 0 \end{cases}$$

Therefore,

$$x(t) = 2x_e(t), \text{ for } t > 0$$

and

$$x(t) = 0, \text{ for } t \leq 0$$

Fourier Transform Tables

Property	Signal	Transform
Linearity	$\alpha\, x_1(t) + \beta\, x_2(t)$	$\alpha\, X_1(j\omega) + \beta \; X_2(j\omega)$
Duality	$X(t)$	$2\pi\, x(-\omega)$
Conjugate symmetry	$x(t)$ real	$X^*(j\omega) = \; X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = \; -\Theta\;(\omega)$ Real part: $X_r(-j\omega) = \; X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = \; -X_i(j\omega)$
Conjugate antisymmetry	$x(t)$ imaginary	$X^*(j\omega) = -X(-j\omega)$ Magnitude: $\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$ Phase: $\Theta(-\omega) = \; -\Theta\;(\omega) \mp \pi$ Real part: $X_r(-j\omega) = \; -X_r(j\omega)$ Imaginary part: $X_i(-j\omega) = \; X_i(j\omega)$
Even signal	$x(-t) = x(t)$	$X(j\omega)$: even
Odd signal	$x(-t) = -x(t)$	$X(j\omega)$: odd
Time shifting	$x(t-\tau)$	$X(j\omega)\, e^{-j\omega\tau}$
Frequency shifting	$x(t)\; e^{j\omega_0 t}$	$X(j(\omega-\omega_0))$
Modulation property	$x(t)\; \cos(\omega_0 t)$	$\frac{1}{2}\; [X(j(\omega-\omega_0)) + \; X(j(\omega+\omega_0))]$
Time and frequency scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}\, X\left(\dfrac{j\omega}{a}\right)$
Differentiation in time	$\dfrac{d^n}{dt^n}\,[x(t)]$	$(j\omega)^n\, X(j\omega)$
Differentiation in frequency	$(-jt)^n\; x(t)$	$\dfrac{d^n}{d\omega^n}\,[X(j\omega)]$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\, X_2(j\omega)$
Multiplication	$x_1(t)\; x_2(t)$	$\dfrac{1}{2\pi}\, X_1(j\omega) * X_2(j\omega)$
Integration	$\displaystyle\int_{-\infty}^{t} x(\lambda)\, d\lambda$	$\dfrac{X(j\omega)}{j\omega} + \pi\, X(0)\, \delta(\omega)$
Parseval's theorem	$\displaystyle\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\; dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$	

Table 4.4 – Fourier transform properties.

Additional properties:		
	x(t): even and real	$X(j\omega)$: even and real
	x(t): odd and real	$X(j\omega)$: odd and imaginary
	x(t): even and imaginary	$X(j\omega)$: even and imaginary
	x(t): odd and imaginary	$X(j\omega)$: odd and real

Name	Signal	Transform
Rectangular pulse	$x(t) = A\,rect(t/\tau)$	$X(j\omega) = A\tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$
Triangular pulse	$x(t) = A\,\Lambda(t/\tau)$	$X(j\omega) = A\tau\,\text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$
Right-sided exponential	$x(t) = e^{-at}\,u(t)$	$X(j\omega) = \frac{1}{a+j\omega}$
Two-sided exponential	$x(t) = e^{-a\|t\|}$	$X(j\omega) = \frac{2a}{a^2+\omega^2}$
Signum function	$x(t) = \text{sgn}(t)$	$X(j\omega) = \frac{2}{j\omega}$
Unit impulse	$x(t) = \delta(t)$	$X(j\omega) = 1$
Sinc function	$x(t) = \text{sinc}(t)$	$X(j\omega) = rect\left(\frac{\omega}{2\pi}\right)$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(j\omega) = 2\pi\,\delta(\omega)$
	$x(t) = \frac{1}{\pi t}$	$X(j\omega) = -j\,\text{sgn}(\omega)$
Unit-step function	$x(t) = u(t)$	$X(j\omega) = \pi\,\delta(\omega) + \frac{1}{j\omega}$
Modulated pulse	$x(t) = rect\left(\frac{t}{\tau}\right)\cos(\omega_0 t)$	$X(j\omega) = \frac{\tau}{2}\,\text{sinc}\left(\frac{(\omega-\omega_0)\tau}{2\pi}\right) + \frac{\tau}{2}\,\text{sinc}\left(\frac{(\omega+\omega_0)\tau}{2\pi}\right)$

Table 4.5 – Some Fourier transform pairs.

Note:

$$\text{sinc}(\alpha) = \frac{\sin(\pi\alpha)}{\pi\alpha}$$

$$\text{rect}(t/\tau) = u(t+\tau/2) - u(t-\tau/2)$$