ECE 102 Midterm

Zheyi Wang

TOTAL POINTS

80.5 / 106

QUESTION 1

Problem 1 19 pts

1.1 a.i 3 / 3

✓ + **3 pts** **Correct (stating this is false) with appropriate justification**

+ **3 pts** Correct (stating this is false) with appropriate justification, but with a minor algebra mistake somewhere (e.g., the integrand should be squared, or other trivial algebra mistake).

+ **2 pts** Correct (stating this is false) with correct calculations, but an incorrect or insufficient justification.

+ **2 pts** Incorrect (stating this is true) due to correct calculations but incorporating a bound. Note, E_y < infinity means it's an energy signal; you want to show E_y is infinite.

+ **1.5 pts** Incorrect (stating that this is true) because of miscalculation of the integral of 1 across all time, but had the problem set up correct, making the correct calculation and indicating what conclusion would have led to the right answer.

+ **0 pts** Incorrect (stating that this is true) or no statement with incorrect or no justification.

1.2 a.ii 3 / 3

✓ + **3 pts** **Correct with appropriate justification.**

+ **0 pts** Incorrect with incorrect justification.

1.3 a.iii 3 / 3

✓ + **3 pts** **Correct with appropriate justification (either through using the properties of an LTI system such as convolution; or stating time invariance preserves the period; or the result from class that complex exponentials are eigenfunctions of LTI systems)**

+ **2 pts** Correct, with a time invariance statement, but that isn't directly related to why this leads to periodicity.

+ **2 pts** Incorrect, but with some work that is almost along the right lines (correct setup) but makes a mistake.

+ **1 pts** Correct, but with inadequate justification (see comments); most common inadequate justification was that this is true because of linearity, but this isn't true, consider y(t) = x(t) * exp(-t).

+ **0 pts** Correct with no justification or a restatement of the question as fact.

+ **0 pts** Incorrect with incorrect justification; or no answer.

1.4 b 9 / 10

✓ + **5 pts** **The system is linear with correct work.**

+ **5 pts** The system is NOT time-invariant with correct work.

✓ + **4 pts** **The system is time-invariant with correct work. (Most often it was not recognizing the t-tau >= 1 statement made the system time-variant, not time-invariant.)**

+ **3 pts** Showed linear, but concluded it was not linear.

+ **2.5 pts** The system is time-invariant without time-shifting the condition (for output) or time-shifting the condition (for input).

+ **2.5 pts** Tried to show system was not linear with counter example; but did algebra incorrectly.

+ **1 pts** Stated definition of linearity; did not show that the system was linear or did so incorrectly.

+ **1 pts** Stated definition of time invariance; did not show the system was time variant.

+ **0 pts** No proof or inadequate proof of linearity; or incorrect statement of what linearity is.

+ **0 pts** No proof of time variance.

QUESTION 2

Problem 2 17 pts

2.1 a 10 / 11

- **0 pts** Correct
- **9 pts** No attempt to find h2(t)

✓ **- 1 pts Did provide the general definition of causality but without specifically explaining it for the given system, or did not use enough/right argument.**

- **2 pts** Did provide the simplified expression of h1(t) but without explaining how
- **5 pts** Incomplete/wrong attempt to find h2(t)
- **3 pts** Did not simplify h1(t)/ wrong h1(t)
- **1 pts** Wrong time argument of one of h2(t) terms
- **2 pts** No or wrong answer for the causality of the system.
- **2 pts** Wrong use of convolutions properties.
- **3 pts** Wrong h2(t) because of wrong h1(t)
- **3 pts** Wrong use of properties of elementary signals
- **1 pts** Wrong copying of h1(t)
- **2 pts** Mathematical mistake when taking the derivative of the integration

2.2 b 6 / 6

✓ **- 0 pts Correct**

- **6 pts** No answer or wrong answer.
- **2 pts** One of the answer term is wrong or incomplete.
- **4 pts** Writing the answer as convolution without further simplification or solving incorrectly the convolution.
- **1 pts** Wrong simplification of x(t)
- **3 pts** Incomplete attempt
- **1 pts** Approach is correct but wrong copying/shifting of function
- **3 pts** Wrong use of shifting or sifting properties
- **2 pts** Convolution with wrong impulse response.

QUESTION 3

Problem 3 16 pts

3.1 a 8 / 8

✓ **- 0 pts Correct**

- **8 pts** No or wrong answer
- **3 pts** Wrong value of b or a.
- **4 pts** Answer with no clear explanation.
- **5 pts** Incomplete attempt.
- **4 pts** Partially correct answer.
- **1 pts** Wrong flipping limits.
- **2 pts** wrong shifting limits

3.2 b 8 / 8

✓ **- 0 pts Correct**

- **5 pts** Missing the sign of the function, or not considering the right shift or flip, (which leads to a wrong answer).
- **1 pts** Wrong value for the area.
- **2 pts** Missing the maximum value.
- **1 pts** Wrong limits of the function after flipping and shifting it.
- **8 pts** No or wrong answer.
- **6 pts** Incomplete answer
- **2 pts** Wrong or missing shifting
- **3 pts** Had the right intuition, but found the wrong t and the wrong area.

QUESTION 4

Problem 4 20 pts

4.1 a 4.5 / 6

- **0 pts** Correct steps toward answer, using Fourier expansion, and proving correct relation between f_k and g_k.
- **4.5 pts** Wrong steps from fk to gk, the integration for Fourier coefficients is written. However, the extracting g_k from integration or definition is not correct

✓ **- 1.5 pts wrong final step/missing final step**

- **0.5 pts** Missing a negative sign
- **2 pts** wrong final presentation
- **4 pts** wrong definition for f_k
- **1 pts** wrong interpretation
- **6 pts** no answer

4.2 b 3 / 6

- **0 pts** Correct
- **6 pts** no answer
- **5 pts** Wrong answer despite correct initial steps

(writing the fourier expansion)

✓ **- 4 pts** **missing final answer with correct path**

- 1 pts missing a negative sign results in odd answers instead of even k s

+ 1 **Point adjustment**

4.3 c.i **1 / 4**

- 0 pts Correct

✓ **- 3 pts** **Wrong answer, writing the answer not in form of Fourier series, wrong attempt to write the even part**

- 2.5 pts correct initial steps but errors leading to incorrect final answer

- 4 pts No answer

- 1 pts minor error in answer (for example wrong negative sign), or missing final representation

4.4 c.ii **2 / 4**

- 0 pts Correct

- 4 pts No answer

✓ **- 3 pts** **Wrong answer, taking initial steps toward answer but missing final answer**

+ 1 **Point adjustment**

QUESTION 5

Problem 5 28 pts

5.1 a **6 / 10**

- 0 pts Correct, writing the correct relation between x(t)/y(t) and comparing output of delayed input and delayed output.

✓ **- 4 pts** **Correct justification of not time-invariance but not complete reasoning (missing the convolution in input/output relation, wrong comparing of two output cases, taking e^{j\omega t} out of convolution, not all correct convolution definition)**

- 7 pts No right format of delayed output and output of delayed input despite writing similar form of convolution, invalid identities leading to wrong forms (Although stating that the e^{j\omega t} is main reason but is not a proof of non-TI)

- 2 pts One part of solution (either the delayed output or output of delayed inout) is missing, wrong definition of convolution

- 10 pts No answer

- 3 pts Wrong format of delaying the convolution, for delaying convolution, just one side of convolution needs to be shifted

- 3 pts Correct answer but Comparing of integrations are not right because two or more functions don't have same format in integration

5.2 b **6.5 / 12**

- 0 pts Correct, considering all parts of system and also reasoning for real and even.

- 12 pts No/ Wrong answer

✓ **- 3.5 pts** **Answer is for only output of h(t) given the input x(t), however the question asks for all system including the multiplier at the beginning, which shifts the X(jw) and therefore the answer is the half left of X(j(w-w0/2))**

- 0.5 pts no reasoning (or wrong) for even/real

✓ **- 2 pts** **not mentioning whether the signal is real/even or not or both wrong answers**

- 4 pts correct path toward H(jw) but not completely right (plotting the x(j(w-w0)), however minor errors leading to wrong answer

- 1 pts wrong acknowledging real or even (one of them)

- 1 pts Minor error in value of Y(iw)

- 2.5 pts Error in sketching Y(jw) given X and H(jw)/ no clear plot for Y(jw)

+ 2 pts partial credit for initial steps toward H(jw)

5.3 C **4 / 6**

- 0 pts Correct

- 6 pts No/Wrong Answer

✓ **- 2 pts** **Wrong power of e (negative sign or the value), the correct power is -jw2/3**

- 2 pts Wrong argument of Y (the correct form is Y(jw/3))

- 5 pts Correct path toward finding shift/scale, however error in steps

- 2 pts Wrong scale factor

- 1 pts wrong sign in either factor or power

QUESTION 6

Problem 6 6 pts

6.1 a 3.5 / 4

+ **4 pts** Correct; the most elegant way was to notice this is a real and even signal multiplied by e^(jt/2) and so the phase is t.

✓ + **3.5 pts** **Very minor mistake on modulation theorem (wrote omega instead of t, or forgot a factor of 2, or both); or other unsimplified answers.**

+ **3 pts** Wrote the signal as a sum of rects and took the inverse Fourier transform, applied modulation correctly, didn't get to the final answer.

+ **2.5 pts** Wrote the signal as a sum of rects and took the inverse Fourier transform, but did not apply the modulation theorem correctly.

+ **2.5 pts** Noticed modulation but did not get answer

+ **2.5 pts** Manually took the inverse Fourier transform but did not extract the phase.

+ **1 pts** Wrote the signal as a sum of rects and recognized the inverse FT would be sincs; or attempted to convolve them.

+ **0 pts** Incorrect or unattempted. (Note, phase of X(jw) is not the phase of x(t)).

6.2 b 0 / 2

+ **2 pts** Correct (or essentially correct)

+ **0.5 pts** Answer does not recognize x(t) = 0 for t < 0.

✓ + **0 pts** **Incorrect or unattempted (even if you wrote the definition of the even component).**

ECE 102, Fall 2018
Department of Electrical and Computer Engineering
University of California, Los Angeles

Midterm
Prof. J.C. Kao
TAs: H. Salami, S. Shahshavari

UCLA True Bruin academic integrity principles apply.
Open: Two pages of cheat sheet allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Wednesday, 14 Nov 2018.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Name: Zheyi Wang

Signature: Zheyi Wang

ID#: 705147852

Problem 1	_____	/	19
Problem 2	_____	/	17
Problem 3	_____	/	16
Problem 4	_____	/	20
Problem 5	_____	/	28
BONUS	_____	/	6 bonus points
Total	_____	/	100 points + 6 bonus points

Problem 1 (19 points)

(a) (9 points) Determine if each of the following statements is true or false. Briefly explain your answer to receive full credit.

i. (3 points) If $x(t)$ is an energy signal, then $y(t) = x(t) + 1$ is also an energy signal.

$$E_y = \lim_{T\to\infty} \int_{-T}^{T} |y(t)|^2 dt$$

$$\therefore \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 - 2|x(t)| + 1 \, dt \le E_y \le \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 + 2|x(t)| + 1 \, dt$$

$$\lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 - 2|x(t)| + 1 \, dt$$

$$= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 - \lim_{T\to\infty} \int_{-T}^{T} 2|x(t)| \, dt + \lim_{T\to\infty} \int_{-T}^{T} 1 \, dt = \infty$$

$\therefore E_y \to \infty$ when $T \to \infty$

$\therefore E_y$ is not an energy signal.

ii. (3 points) If $x(t)$ is an even signal, then $y(t) = x(t-1)$ is also an even signal.

counterexample. $x(t) = \cos t$

$y(t) = \cos(t-1)$

$y(1) \neq y(-1)$

isn't even

iii. (3 points) If the input to an LTI system is periodic, then its output is also periodic.

by convolution TI property

$y(t-\tau) = x(t-\tau) * h(t)$, Let T be the period of $x(t)$

$\therefore y(t-T) = x(t-T) * h(t)$

$= x(t) * h(t) = y(t)$

$\therefore$ y is also periodic.

(b) (10 points) Is the following system linear? Is it time invariant? (Check both properties). Explain your answer.

$$y(t) = \begin{cases} x(t-1), & t \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

(1) $y(t) = S(x(t))$

$$S(ax_1(t) + bx_2(t)) = \begin{cases} ax_1(t-1) + bx_2(t-1) & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$aS(x_1(t)) + bS(x_2(t)) = \begin{cases} ax_1(t-1) + bx_2(t-1) & t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore aS(x_1(t)) + bS(x_2(t)) = S(ax_1(t) + bx_2(t))$

$\therefore$ linear.

(2) $$S(x(t-\tau)) = \begin{cases} x(t-\tau-1) & t \geq 1+\tau \\ 0 & \text{otherwise} \end{cases}$$

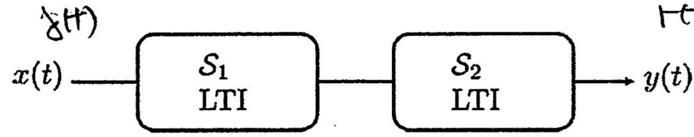
$$y(t-\tau) = \begin{cases} x(t-\tau-1) & t-\tau \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore S(x(t-\tau)) = y(t-\tau)$

$\therefore$ TI

Problem 2 (17 points) Consider the series cascade of the following two systems:

$\delta(t)$ $\quad$ $r(t-3)+u(t-2)$



$X(t) * h_1(t) * h_2(t) = y(t)$

The system S_1 is LTI with impulse response

$$h_1(t) = \int_{-\infty}^{t} u(\tau)\delta(\tau-2)d\tau = u(2)\int_{-\infty}^{t}\delta(\tau-2)d\tau = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 2 \end{cases}$$

The system S_2 is also LTI, with unknown impulse response $h_2(t)$ that we need to find. We are $u(t-2)$ also given that, when the input $x(t)$ is $\delta(t)$, the output $y(t)$ is $r(t-3)+u(t-2)$.

Note: $r(t-3)$ is the ramp function delayed by 3.

This question continues on the next page.

(a) (11 points) Find the impulse response $h_2(t)$ of the system S_2 and determine if the system S_2 is causal.

$$h_1(t) = \int_{-\infty}^{t} u(\tau)\,\delta(\tau-2)\,d\tau$$

$$= \int_{-\infty}^{t} u(2)\,\delta(\tau-2)\,d\tau$$

$$= u(2)\int_{-\infty}^{t} \delta(\tau-2)\,d\tau$$

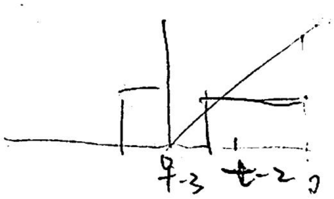
$$= u(t-2)$$

$$\therefore \quad x(t) * h_1(t) = u(t-2)$$

$$u(t-2) * h_2(t) = r(t-3) + u(t-2)$$

$$\therefore \quad \int_{-\infty}^{\infty} h_2(\tau)\cdot u(t-2-\tau)\,d\tau = u(t-2) + r(t-3)$$

$$\int_{-\infty}^{t-2} h_2(\tau)\,d\tau = u(t-2) + r(t-3)$$



$$h_2(t) = \delta(t) + u(t-1)$$

casual since only present and past needed

$t-3 > 0.$

(b) (6 points) Find the output $y(t)$ to the following input:

$$x(t) = (1 + e^{-t})\delta(t + 1) = (1+e^{-1})\delta(t+1)$$

$y(t) = h(t) * x(t)$

$h(t) = r(t-3) + u(t-2)$ $\because y(t) = h(t)$ when $x(t) = \delta(t)$

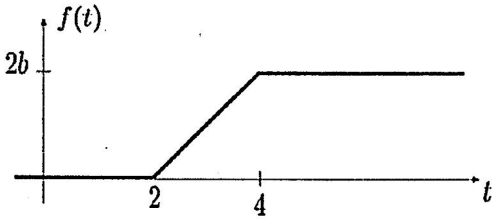
$\therefore y(t) = r(t-3) * (1+e^{-1})\delta(t+1) + u(t-2) * (1+e^{-1})\delta(t+1)$

$= (1+e^{-1})\left(\left[r(t-3) * \delta(t+1)\right] + \left[u(t-2) * \delta(t+1)\right]\right)$

$= (1+e^{-1})\left(r(t-2) + u(t-1)\right)$

Problem 3 (16 points)

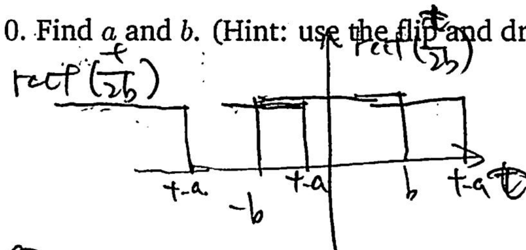
(a) (8 points) Consider the signal $f(t)$ shown below:



This signal can be written as

$$u(t-a) * \mathrm{rect}\left(\frac{t}{2b}\right)$$

where $a > 0$ and $b > 0$. Find a and b. (Hint: use the flip and drag technique.)



$$\int_{-\infty}^{\infty} u(t-a-\tau)\,\mathrm{rect}\left(\frac{\tau}{2b}\right) d\tau$$

$t-a<-b$ $\quad \int_{-\infty}^{\infty} u(t-a-\tau)\,\mathrm{rect}\left(\frac{\tau}{2b}\right) d\tau = 0$

$-b<t-a<b$ $\quad \int_{-\infty}^{\infty} u(t-a-\tau)\,\mathrm{rect}\left(\frac{\tau}{2b}\right) d\tau = t-a+b$

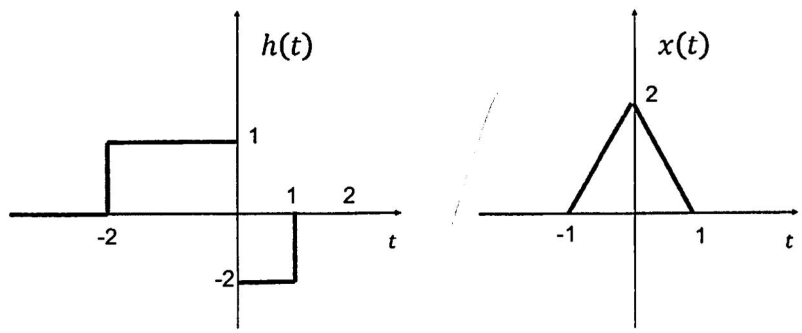
$t-a>b$ $\quad \int_{-\infty}^{\infty} u(t-a-\tau)\,\mathrm{rect}\left(\frac{\tau}{2b}\right) d\tau = 2b$

$$u(t-a) * \mathrm{rect}\left(\frac{t}{2b}\right) = \begin{cases} 0 & t<a-b \\ t-a+b & t\in(a-b,a+b) \\ 2b & t>a+b \end{cases}$$

$\therefore 2b = 4-2 \Rightarrow b=1$

$a-b=2 \Rightarrow a=3$

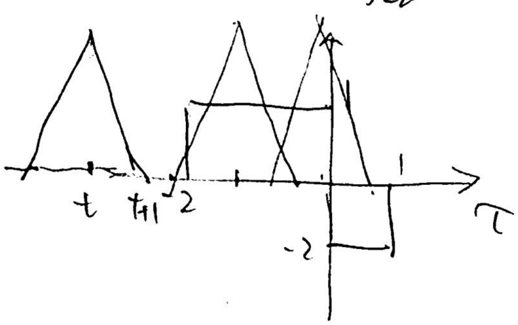
(b) (8 points) An input, $x(t)$, is given to an LTI system with impulse response $h(t)$. Both $x(t)$ and $h(t)$ are shown below.



Let $y(t)$ denote the output of the system, i.e., $y(t) = x(t) * h(t)$. Find the value of t at which the output $y(t)$ reaches its maximum value. Determine this maximum value.

Note: to answer this question, you do **not** *need to find $y(t)$ for all t.*

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



when t increa

$t<-3$	$y(t)=0$	
$t \in (-3,-1)$	$y(t)$ increase.	~
$t \in (-1,0)$	$y(t)$ decrease	~
$t \in (0,1)$	$y(t)$ decrease	~
$t \in (1,2)$	$y(t)$ decrease	~
$t \in (2,\infty)$	$y(t)$ not change.	~

y_{max} when $t=-1$ $\therefore y(t)_{max} = y(-1) = \int_{-\infty}^{\infty} h(\tau)\, x(-1-\tau)\, d\tau$

$$= \int_{-2}^{0} x(\tau+1)\, d\tau$$

$$= 2$$

Problem 4 (20 points)

Consider the following two periodic signals $f(t)$ and $g(t)$. They both have the same period T_0. Let f_k and g_k respectively denote the Fourier series coefficients of $f(t)$ and $g(t)$.

(a) (6 points) If $f(t) = -g\left(t + \frac{T_0}{2}\right)$, how is f_k related to g_k?

$$f_k = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} -g(t+\frac{T_0}{2})\, e^{-jw_0tk}\, dt$$

$$t' = t + \frac{T_0}{2}$$

$$f_k = \frac{1}{T_0}\int_{\tau'}^{\tau'+T_0} -g(t')\, e^{-jw_0t'k + jw_0\frac{T_0}{2}k}\, dt'$$

$$= -e^{2\pi jk}\frac{1}{T_0}\int_{\tau'}^{\tau'+T_0} g(t')\, e^{-jw_0t'k}\, dt'$$

$$= -e^{2\pi jk}\cdot g_k$$

$$= -\cos(2\pi k)\, g_k$$

$$= -g_k.$$

(b) (6 points) If $f(t) = -f\left(t + \frac{T_0}{2}\right)$, for what k are the coefficients f_k zero?

$$f_k = \frac{1}{T_0} \int_{\tau}^{\tau + T_0} - f\left(t + \frac{T_0}{2}\right) e^{-j\omega_0 t k} \, dt$$

$$= -e^{2\pi j k} f_k$$

$$= -\left(\cos(2\pi k) f_k + j\sin(2\pi k) f_k\right)$$

$$= -\cos(2\pi k) f_k = -f_k$$

$f_k + f_k = 0 \quad \therefore \quad f_k = 0$

(c) (8 points) This question has two parts. *Note: part (c) is independent of parts (a) and (b).*

i. (4 points) Let $f_e(t)$ denote the even part of $f(t)$. Express the Fourier series coefficients of $f_e(t)$ in terms of f_k.

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \, e^{jk\omega_0 t}$$

$$f(-t) = \sum_{k=-\infty}^{\infty} f_k \, e^{-jk\omega_0 t} = \sum_{k'=-\infty}^{\infty} f_{-k} \, e^{jk\omega_0 t}$$

$$\therefore f_e(t) = \frac{1}{2} f(t) + \frac{1}{2} f(-t)$$

$$= \sum_{k=-\infty}^{\infty} f_k \left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right) \cdot \frac{1}{2}$$

$$= \sum_{k=-\infty}^{\infty} f_k \cdot \cos(k\omega_0 t)$$

ii. (4 points) Determine the DC component of $f_o(t)$, the odd part of $f(t)$.

$$f_o(t) = \sum_{k=-\infty}^{\infty} f_k \left(e^{jk\omega_0 t} - e^{-jk\omega_0 t}\right)$$

$$= \sum_{k=-\infty}^{\infty} f_k \; j\sin(k\omega_0 t)$$

$$= f_0 + \sum_{k=-\infty}^{-1} f_k \; j\sin(k\omega_0 t) + \sum_{k=1}^{\infty} f_k \; j\sin(k\omega_0 t)$$

$$= f_0 + \sum_{k=1}^{\infty} -f_{-k} \; j\sin(k\omega_0 t) + \sum_{k=1}^{\infty} f_k \, j\sin(k\omega_0 t)$$

$$= f_0 + \sum_{k=1}^{\infty} (f_k - f_{-k}) \; j\sin(k\omega_0 t)$$

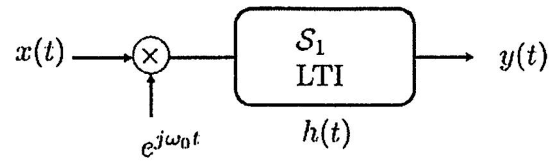
for $f(t)$ odd $\quad f_k = -f_{-k}$ from class

$$\therefore f_o(t) = f_0 + \sum_{k=1}^{\infty} 2 f_k \, j\sin(k\omega_0 t)$$

DC component $\sum_{k=1}^{\infty} 2 f_k \, j\sin(k\omega_0 t)$

Problem 5 (28 points)

Consider the following system ($\omega_0 > 0$):



The system S_1 is LTI and $h(t)$ represents its impulse response.

(a) (10 points) Show that the overall system, with input $x(t)$ and output $y(t)$, is not time-invariant.

$$S(x(t)) = h(t) * \left(x(t)\, e^{j\omega_0 t}\right)$$

$$S(x(t-\tau)) = h(t) * x(t-\tau)\, e^{j\omega_0 t}$$

$$y(t-\tau) = h(t) * x(t-\tau)\, e^{j\omega_0 t}$$

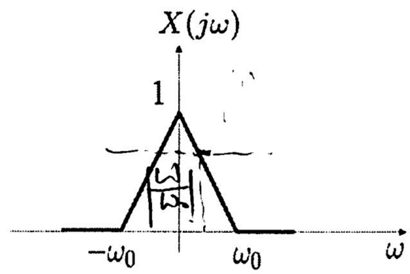
$$y(t-\tau) \neq S(x(t-\tau))$$

$\therefore$ not TI

(b) (12 points) Consider the following impulse response for system $\mathcal{S}_1$:

$$h(t) = e^{j\frac{\omega_0}{2}t}\text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$$

We give the system an input $x(t)$, where $x(t)$ has the following Fourier transform $X(j\omega)$:



Find and sketch the Fourier transform $Y(j\omega)$ of the corresponding output $y(t)$. After this, determine (i) if $y(t)$ is real and (ii) if $y(t)$ is even. *Note: you do not need to give an expression for $Y(j\omega)$, a sketch of it is enough. There is space on the next page if needed.*

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

Let $g(t) = \text{sinc}\left(\frac{\omega_0}{2\pi}t\right)$

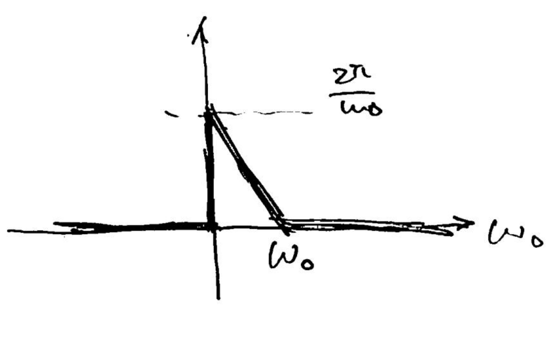
$$G(j\omega) = \frac{2\pi}{\omega_0}\,\text{rect}\left(\frac{\omega}{2\pi}\Big/\frac{\omega_0}{2\pi}\right)$$

$$= \frac{2\pi}{\omega_0}\,\text{rect}\left(\frac{\omega}{\omega_0}\right)$$

$$h(t) = g(t)\,e^{j\frac{\omega_0}{2}t}$$

$$\therefore H(j\omega) = G\left(j\left(\omega - \tfrac{1}{2}\omega_0\right)\right) = \frac{2\pi}{\omega_0}\,\text{rect}\left(\frac{\omega}{\omega_0} - \frac{1}{2}\right)$$

$$Y(j\omega) = \begin{cases} \left|\frac{\omega}{\omega_0}\right| \cdot \frac{2\pi}{\omega_0}\,\text{rect}\left(\frac{\omega}{\omega_0} - \frac{1}{2}\right) & \omega \le |\omega_0| \\ 0 & \text{otherwise} \end{cases}$$



(c) (6 points) Suppose

$$z(t) = y(3t - 2)$$

Express $Z(j\omega)$ in terms of $Y(j\omega)$. *Note: part (c) is independent of parts (a) and (b).*

$z(t) = y\left(3\left(t - \frac{2}{3}\right)\right) = x(3t) \quad x(t) = y\left(t - \frac{2}{3}\right)$

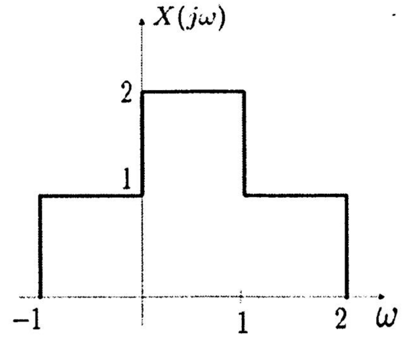
$Z(j\omega) = \frac{1}{3} X\left(j\frac{\omega}{3}\right)$

$X(j\omega) = Y(j\omega)\, e^{-\frac{2}{3}j\omega}.$

$\therefore \quad Z(j\omega) = \frac{1}{3} Y\left(\frac{j\omega}{3}\right) e^{-\frac{2}{9}j\omega}$

BONUS (6 points)

(a) (4 points) The Fourier transform $X(j\omega)$ of a signal $x(t)$ is given as follows:



Find the phase of $x^2(t)$.

$$X(j\omega) = \mathrm{rect}\left(\frac{\omega-\frac{1}{2}}{3}\right) + \mathrm{rect}\left(\omega-\tfrac{1}{2}\right)$$

$$X(t) = \mathcal{F}\left(\mathrm{rect}\left(\frac{\omega-\frac{1}{2}}{3}\right)\right) + \mathcal{F}\left(\mathrm{rect}\left(\omega-\tfrac{1}{2}\right)\right)$$

$$= 3\,\mathrm{sinc}\frac{3t}{2\pi}\, e^{-\frac{3}{2}jt} + \mathrm{sinc}\frac{t}{2\pi}\, e^{-\frac{1}{2}jt}$$

$Re(X(t))$

$$= \left(3\,\mathrm{sinc}\frac{3t}{2\pi}\cos\left(\tfrac{3}{2}t\right) + \mathrm{sinc}\frac{t}{2\pi}\cos\left(\tfrac{1}{2}t\right)\right)$$

$Im(X(t))$

$$+ \left(3\,\mathrm{sinc}\frac{3t}{2\pi}\sin\left(\tfrac{3}{2}t\right) + \mathrm{sinc}\frac{t}{2\pi}\sin\left(\tfrac{1}{2}t\right)\right)j$$

$$X(t)^2 = Re^2(X(t)) + 2Re(X(t))Im(X(t))j - Im^2(X(t))$$

$$\therefore \angle X(t)^2 = \frac{2\,Re(X(t))\,Im(X(t))}{Re^2(X(t))\ \ Im^2(X(t))}$$

$$= \boxed{\frac{2}{\frac{1}{\angle X(t)} - \angle X(t)}}$$

for which
$$\angle X(t) = \frac{3\,\frac{\left(\sin\frac{3}{2}t\right)^2}{\frac{3}{2}t} + \frac{\left(\sin\frac{1}{2}t\right)^2}{\frac{1}{2}t}}{3\,\frac{\sin\frac{3}{2}t\cos\frac{3}{2}t}{\frac{3}{2}t} + \frac{\sin\frac{1}{2}t\cos\frac{1}{2}t}{\frac{1}{2}t}}$$

$$= \frac{\sin^2\frac{3}{2}t + \sin^2\frac{1}{2}t}{\sin\frac{3}{2}t\cos\frac{3}{2}t + \sin\frac{1}{2}t\cos\frac{1}{2}t}$$

(b) (2 points) If a signal $x(t)$ is causal with $x(0) = 0$, how can we retrieve $x(t)$ from its even component $x_e(t)$?

$$X_e(t) = \frac{1}{2}\left(X(t) + X(-t)\right)$$

$$X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, d\omega \qquad X(j\omega) \text{ is odd}$$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{-j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)\cancel{X}\cos(\omega t)}_{\text{odd}} - jX(j\omega)\sin(\omega t)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -jX(j\omega)\sin(\omega t)\, d\omega$$

$X(t)$ is odd