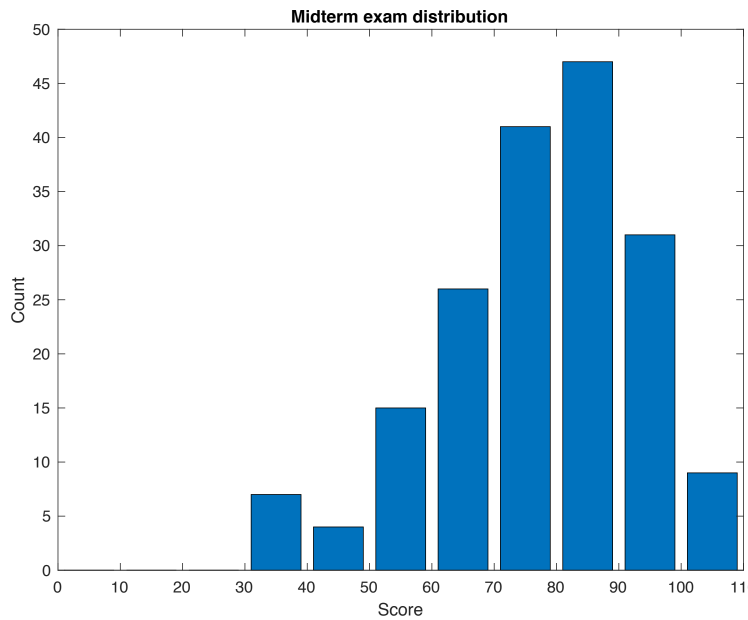
ECE 102, Fall 2020 **MIDTERM STATISTICS**
Department of Electrical and Computer Engineering Prof. J.C. Kao
University of California, Los Angeles TAs A. Ghosh, T. Monsoor & G. Zhao



Statistics: Mean: 76.3 (out of 100), Standard deviation: 15.9, Median: 78.5, Maximum score: 106, Number of exams: 180

Comments:

- If you have grading questions, please submit through Gradescope. Regrades should *only be submitted* if you believe we applied the rubric mistakenly. When we regrade, we re-evaluate the entire question, and so while rare, losing points is possible if we mis-graded your work.
- If you have particular questions about a midterm question and its grading, please see: Q1: Guangyuan, Q2: Tonmoy, Q3: Arunabh, Bonus: Jonathan
- The variance of the midterm was higher than we expected. We were pleased that many students did solidly on the exam, however, we recognize that many students did not do as well.

  We are therefore making you the following offer: if you score better on the final than the midterm, we will replace your midterm score with your final score.

ECE102, Fall 2020 **Midterm Solution**

Department of Electrical and Computer Engineering Prof. J.C. Kao
University of California, Los Angeles TAs: A. Ghosh, T. Monsoor & G. Zhao

UCLA True Bruin academic integrity principles apply.
Open: Two cheat sheets allowed.
Closed: Book, computer, internet.
2:00-3:50pm.
Monday, 9 Nov 2020.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Do not send a copy of this exam to any classmates or discuss the exam on Piazza until after Wednesday, 11 Nov 2020. There are students who will take the exam at different times.

Name: ____________________

Signature: ____________________

ID#: ____________________

Problem 1	____	/ 30
Problem 2	____	/ 40
Problem 3	____	/ 30
BONUS	____	/ 6 bonus points
Total	____	/ 100 points + 6 bonus points

1. **Signal and System Properties** (30 points).

(a) (6 points) *Complex number.* What are the phase and amplitude of the following number?

$$x = (1+j)e^{3j} \tag{1}$$

Solution:

$$x = \sqrt{2}e^{j\frac{\pi}{4}}e^{3j} = \sqrt{2}e^{(\frac{\pi}{4}+3)j}$$

Hence, the amplitude is $\sqrt{2}$, the phase is $3 + \frac{\pi}{4}$

(b) (6 points) If $x(t)$ is an even function, and $x(t-1)$ is also even, is $x(t)$ periodic? Explain your reasoning.

Solution:
As $x(t)$ and $x(t-1)$ are both even, we have $x(t) = x(-t)$ and $x(t-1) = x(-t-1)$. If we substitute the $t-1$ with t', we have $x(-t') = x(t') = x(-t-1) = x(-(t-1)-2) = x(-t'-2)$. Thus we can see that $x(t)$ is periodic.

(c) (6 points) Evaluate the expression

$$\int_0^{\infty} \delta(t-1)(t-2)^2 dt.$$

Solution:
Recall that according to the **sifting property**,

$$f(t)\delta(t-1) = f(1)\delta(t-1).$$

In this case, $f(t) = (t-2)^2$; therefore,

$$(1-2)^2\delta(t-1) = (1)^2\delta(t-1) = \delta(t-1).$$

Finally, we integrate the above expression,

$$\int_0^{\infty} (t-2)^2\delta(t-1)dt = 1\int_0^{\infty} \delta(t-1) = 1.$$

(d) (12 points) Consider the system with input $x(t)$ and output $y(t)$:

$$y(t) = \int_{-\infty}^{t} x(\lambda)u(t+1)d\lambda.$$

Determine if the system is:

i. Linear

ii. Time-invariant

iii. Causal

iv. Stable

You must justify your answer to receive full credit.

Solution:

Linearity: **Linear** We will check here homogeneity and superposition:

$$
\begin{aligned}
u(t+1)\int_{-\infty}^{t}(ax_1(\lambda)+bx_2(\lambda))\,d\lambda &= u(t+1)\int_{-\infty}^{t}ax_1(\lambda)d\lambda + u(t+1)\int_{-\infty}^{t}bx_2(\lambda)d\lambda \\
&= a\left(u(t+1)\int_{-\infty}^{t}x_1(\lambda)d\lambda\right)+b\left(u(t+1)\int_{-\infty}^{t}x_2(\lambda)d\lambda\right) \\
&= ay_1(t)+by_2(t)
\end{aligned}
$$

The system is then linear.

Time-invariance: **Time-varint** If we delay the input by τ, i.e., $x_\tau(t) = x(t-\tau)$, the output is:

$$y_\tau(t) = u(t+1)\int_{-\infty}^{t}x_\tau(\lambda)d\lambda = u(t+1)\int_{-\infty}^{t}x(\lambda-\tau)d\lambda$$

Let $\lambda' = \lambda - \tau$, then

$$y_\tau(t) = u(t+1)\int_{-\infty}^{t-\tau}x(\lambda')d\lambda'$$

On the other hand,

$$y(t-\tau) = u(t+1-\tau)\int_{-\infty}^{t-\tau}x(\lambda)d\lambda$$

Therefore $y(t-\tau) \neq y_\tau(t)$. The system is then time variant.

Causality: **Causal** The system is integrating values of $x(t)$ up to time t. The output does not depend on future values of $x(t)$, the system is then causal.

Stability: **Unstable** Even if $x(t)$ is absolutely bounded, the integral:

$$\int_{-\infty}^{t}x(\lambda)d\lambda$$

cannot in general be bounded, the system is unstable. For instance, suppose $x(t) = 1$, then $\int_{-\infty}^{t}1d\lambda \to \infty$. Another example, suppose $x(t) = u(t)$, then

$$y(t) = u(t+1)\int_{-\infty}^{t}u(\lambda)d\lambda = u(t+1)\int_{0}^{t}1d\lambda = u(t+1)t$$

$u(t+1)t$ cannot be bounded as $t \to \infty$, because $u(t+1)t \to \infty$ as $t \to \infty$.

2. **Impulse response, LTI systems and Convolution** (40 points).

Note: Parts (a) and (b) of this problem are independent. In part (a) of the problem, you might find the following convolution result useful

$$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

(a) Consider the LTI system characterized by the Input/Output relationship:

$$\text{System 1 :} \qquad y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}x(\tau)d\tau$$

i. (5 points) Write the impulse response of the system, $h_1(t)$.

Solution: Impulse response is the output of the system when the input is the dirac delta, $\delta(t)$

$$h_1(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}\delta(\tau)d\tau$$

Using the sifting property of the impulse,

$$h_1(t) = e^{-2t}\int_{-\infty}^{t} \delta(\tau)d\tau$$

$$h_1(t) = e^{-2t}u(t)$$

ii. (10 points) Determine a closed-form expression for the output (y(t)) of System 1 for the input

$$x(t) = 3e^{-4t}u(t-2)$$

Solution: Let's first compute the output of the system when the input is $e^{-4t}u(t)$

$$y_1(t) = e^{-4t}u(t) * e^{-2t}u(t)$$

Using the convolution result,

$$y_1(t) = -\frac{1}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t)$$

We can rewrite the input $x(t)$ as follows

$$x(t) = 3e^{-4(t-2)}u(t-2) \times e^{-8}$$

Then using the linearity and time-invariance property of system 1,

$$\begin{aligned} y(t) &= 3e^{-8}y_1(t-2) \\ &= 3e^{-8}\{-\frac{1}{2}e^{-4(t-2)}u(t-2) + \frac{1}{2}e^{-2(t-2)}u(t-2)\} \end{aligned}$$

iii. (10 points) Determine a closed-form expression for the output (y(t)) of System 1 for the input

$$x(t) = \{u(t) - u(t-3)\}$$

Solution: The output of an LTI system is given by the convolution

$$\begin{aligned} y(t) &= \{u(t) - u(t-3)\} * e^{-2t}u(t) \\ &= e^{-2t}u(t) * u(t) - e^{-2t}u(t) * u(t-3) \end{aligned}$$

Using the convolution result,

$$\begin{aligned} y_2(t) &= e^{-2t}u(t) * u(t) \\ &= -\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) \end{aligned}$$

Since the system is time-invariant,

$$\begin{aligned} y(t) &= y_2(t) - y_2(t-3) \\ &= \{-\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t)\} - \{-\frac{1}{2}e^{-2(t-3)}u(t-3) + \frac{1}{2}u(t-3)\} \end{aligned}$$

iv. (5 points) Determine the output of System 1, y(t), for the input

$$x(t) = \delta(t) - \delta(t-3)$$

Solution: We know that the output of an LTI system is given by the convolution of the input and the impulse response

$$\begin{aligned}
y(t) &= x(t) * h_1(t) \\
&= \{\delta(t) - \delta(t-3)\} * h_1(t) \\
&= \{\delta(t) * h_1(t\}) - \{\delta(t-3) * h_1(t)\} \\
&= h_1(t) - h_1(t-3) \\
y(t) &= e^{-2t}u(t) - e^{-2(t-3)}u(t-3)
\end{aligned}$$

(b) If $y(t) = x(t) * h(t)$ is the output of an LTI system with input $x(t)$ and impulse response $h(t)$, then show the following properties. Note, you will not receive credit if you simply state that the differentiator and convolution are LTI systems and can be interchanged. Although this is true, we want you to work with the convolution integral.

Hint: You may interchange the order of integration and differentiation.

i. (5 points) $\frac{d}{dt}y(t) = x(t) * (\frac{d}{dt}h(t))$

Solution: We know that the output of the LTI system is given by

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau
\end{aligned}$$

Now, taking the derivative with respect to t of both sides

$$\frac{d}{dt}y(t) = \frac{d}{dt}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Assuming that the functions are sufficiently smooth, the derivative can be pulled through the integral

$$\begin{aligned}
\frac{d}{dt}y(t) &= \int_{-\infty}^{\infty} \frac{d}{dt}\{x(\tau)h(t-\tau)\}d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\frac{d}{dt}h(t-\tau)d\tau \\
\frac{d}{dt}y(t) &= x(t) * (\frac{d}{dt}h(t))
\end{aligned}$$

ii. (5 points) $\frac{d}{dt}y(t) = (\frac{d}{dt}x(t)) * h(t)$

Solution: Due to the commutative property of convolution,

$$\begin{aligned}
y(t) &= h(t) * x(t) \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau
\end{aligned}$$

Now, taking the derivative with respect to t of both sides

$$\frac{d}{dt}y(t) = \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Assuming that the functions are sufficiently smooth, the derivative can be pulled through the integral

$$\begin{aligned}
\frac{d}{dt}y(t) &= \int_{-\infty}^{\infty} \frac{d}{dt}\{h(\tau)x(t-\tau)\}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau \\
\frac{d}{dt}y(t) &= h(t) * (\frac{d}{dt}x(t)) = (\frac{d}{dt}x(t)) * h(t)
\end{aligned}$$

3. **Fourier Series** (30 points).

(a) (15 points) Find the Fourier series of the function: $f(x) = \begin{cases} -1 & \text{if } -\pi \le x \le 0 \\ 1 & \text{if } 0 < x \le \pi \end{cases}$. To receive full credit, you must simplify your answer, including complex exponentials if possible.

Solution:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = \frac{1}{2\pi}\left[(-x)|_{-\pi}^{0} + x|_{0}^{\pi}\right] = \frac{1}{2\pi}(-\pi+\pi) = 0$$

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-jnx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,e^{-jnx}dx + \int_{0}^{\pi} e^{-jnx}dx\right] \\ &= \frac{1}{2\pi}\left[-\frac{\left(e^{-jnx}\right)|_{-\pi}^{0}}{-jn} + \frac{\left(e^{-jnx}\right)|_{0}^{\pi}}{-jn}\right] = \frac{j}{2\pi n}\left[-\left(1-e^{jn\pi}\right) + e^{-jn\pi}-1\right] = \frac{j}{2\pi n}\left[e^{jn\pi} + e^{-jn\pi}-2\right] \\ &= \frac{j}{\pi n}\left[\frac{e^{jn\pi}+e^{-jn\pi}}{2}-1\right] = \frac{j}{\pi n}\left[\cos n\pi-1\right] = \frac{j}{\pi n}\left[(-1)^n-1\right]. \end{aligned}$$

If $n = 2k$, then $c_{2k} = 0$. If $n = 2k-1$, then $c_{2k-1} = -\frac{2i}{(2k-1)\pi}$.
Hence, the Fourier series of the function is:

$$f(x) = \operatorname{sign} x = -\frac{2i}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{2k-1}e^{i(2k-1)x}.$$

(b) (15 points) Let $x(t)$ be a periodic signal whose Fourier series coefficients are:

$$c_k = \begin{cases} 2 & k = 0 \\ j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$$

Use Fourier series properties to answer the following questions:

i. Is $x(t)$ real?
ii. Is $x(t)$ even?
iii. Is $\frac{dx(t)}{dt}$ even?

Solution:

i. We know that if $x(t)$ is real, then $c_k = c^*_{-k}$. We can see that,

$$c^*_{-k} = \begin{cases} 2 & k = 0 \\ -j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$$

Therefore we can see that $c_k \neq c^*_{-k}$. $x(t)$ is not real.

ii. We know that if $x(t)$ is even, then $c_k = c_{-k}$. We can see that,

$c_{-k} = \begin{cases} 2 & k = 0 \\ j(\frac{1}{2})^{|k|} & \text{otherwise} \end{cases}$

Therefore we can see that $c_k = c_{-k}$. $x(t)$ is even.

iii. We know that,

$$x(t) = \sum_{k=\infty}^{\infty} c_k e^{jk\omega_0 t}$$

then,

$$\frac{dx(t)}{dt} = \sum_{k=\infty}^{\infty} c_k jk\omega_0 e^{jk\omega_0 t}$$

From this we can see that, if b_k denotes the Fourier coefficients of $\frac{dx(t)}{dt}$ then $b_k = c_k jk\omega_0$. So,

$b_k = \begin{cases} 0 & k = 0 \\ -j(\frac{1}{2})^{|k|} k\omega_0 & \text{otherwise} \end{cases}$

From this we can see that $\frac{dx(t)}{dt}$ is not even.

Bonus Question (6 points)

Consider two signals, $x(t) = \text{rect}(t - 2.5)$ and $h(t) = \text{rect}(t)$. We convolve $x(2t)$ and $h(2t)$, i.e.,

$$y(t) = x(2t) * h(2t)$$

Using the flip and drag technique, compute the values of:

- $y(0.5)$
- $y(1)$

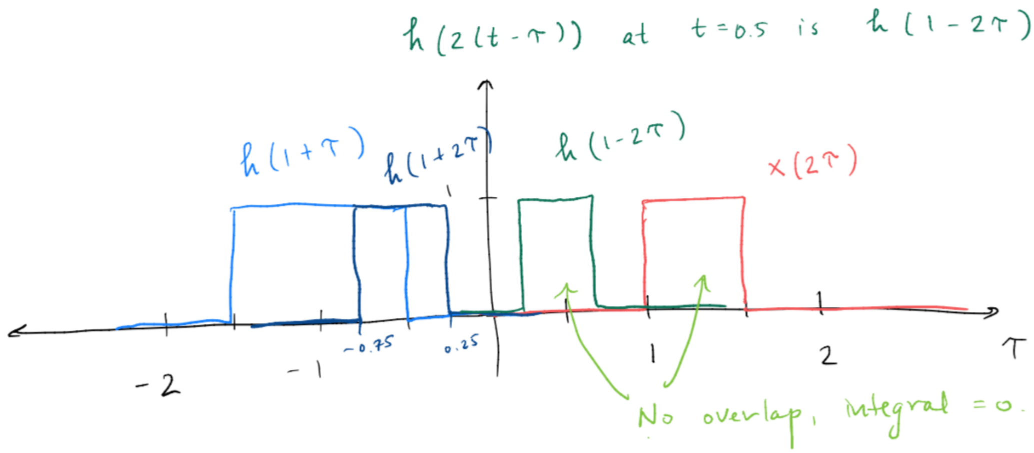
To receive full credit, you must draw a sketch of the flip and drag convolution computation you use to compute $y(0.5)$ and $y(1)$. You will not receive any partial credit for solving this analytically, although you may of course use an analytical answer to check your work. (This is because the purpose of this question is to test your understanding of signal operations and computing convolution.)

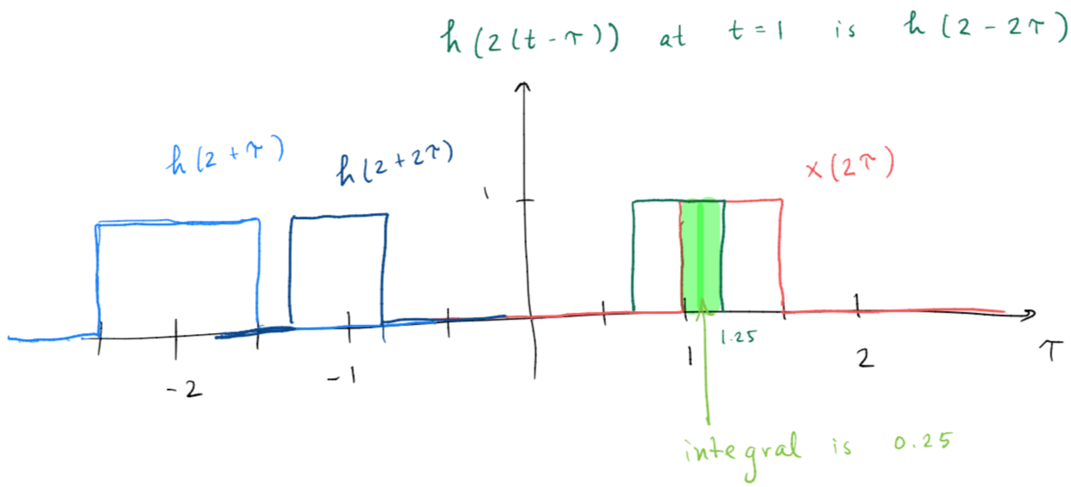
Solution:

The convolution is:

$$y(t) = \int_{-\infty}^{\infty} x(2\tau)h(2(t - \tau))d\tau$$

In the flip and drag, we choose to flip $h(\cdot)$. Then, $x(2\tau)$ is a rect that between $t = 1$ and $t = 1.5$. Below, we illustrate where $h(2(t - \tau))$ is at $t = 0.5$ and $t = 1$. These show that $y(0.5) = 0$ and $y(1) = 0.25$.


h(2(t-τ)) at t=0.5 is h(1-2τ)
h(1+τ)
h(1+2τ)
h(1-2τ)
x(2τ)
1
-0.75
0.25
-2
-1
1
2
τ
No overlap, integral = 0.


h(2(t-τ)) at t=1 is h(2-2τ)
h(2+τ)
h(2+2τ)
x(2τ)
1
1.25
-2
-1
1
2
τ
integral is 0.25

ECE 102 Midterm

[redacted]

1. (a) $x = (1+j)e^{3j}$

$1+j \Rightarrow$ magnitude $\sqrt{1^2+1^2} = \sqrt{2}$

phase $\arctan(\frac{1}{1}) = \pi/4$

so $x = \sqrt{2}e^{j\pi/4} \cdot e^{3j}$

$x = \sqrt{2}e^{j(\frac{\pi}{4}+3)}$

$|x| = \sqrt{2}$

$\measuredangle x = \frac{\pi}{4}+3$

(b) If $x(t)$ is even then $x(t-1)$ does not have to be even but since it is $x(t)$ must be periodic with period 2, since even properties can be used to show

$x(t) = x(-t) : x(-2) = x(2)$

$x(-t-1) = x(t-1) : x(-2) = x(0)$ } Period 2.

We can extend this for all t and show $x(t) = x(t+2)$ thus $x(t)$ is periodic

(c) Evaluate

$$\int_0^\infty \delta(t-1)(t-2)^2 dt$$

By sampling property: $\int_0^\infty \delta(t-1)(1-2)^2 dt$

$$= \int_0^\infty \delta(t-1) = \boxed{1}$$

(d) (i) Linearity Check:

Input $ax_1(\lambda)+bx_2(\lambda) \Rightarrow \int_{-\infty}^{t}(ax_1(\lambda)+bx_2(\lambda))u(t+1)\,d\lambda$

$$= \int_{-\infty}^{t} ax_1(\lambda)u(t+1)\,d\lambda + \int_{-\infty}^{t} bx_2(\lambda)u(t+1)\,d\lambda$$

$$= a\int_{-\infty}^{t} x_1(\lambda)u(t+1)\,d\lambda + b\int_{-\infty}^{t} x_2(\lambda)u(t+1)\,d\lambda$$

Which equals $ay_1(t) + by_2(t)$. Thus it **IS Linear**

(ii) Time-Invariant:

Delay Input: $x(\lambda) = x(\lambda - \tau) \Rightarrow \int_{-\infty}^{t} x(\lambda - \tau)\, u(t+1)\, d\lambda$

Delay Output: $y(t) = y(t - \tau) \Rightarrow \int_{-\infty}^{t-\tau} x(\lambda)\, u(t - \tau + 1)\, d\lambda$

Let $\lambda' = \lambda + \tau \Rightarrow \int_{-\infty}^{t} x(\lambda' - \tau)\, u(t - \tau + 1)\, d\lambda'$

NOT SAME

Not matching, thus **NOT Time Invariant**

(iii) Causal:

$y(t) = \int_{-\infty}^{t} x(\lambda)\, u(t+1)\, d\lambda$

Example $t = 1$: $y(1) = \int_{-\infty}^{1} x(\lambda)\, u(2)\, d\lambda$

$u(2)$ makes use of future value thus, **NOT Causal**

(iv) Stability:

Assume $|x(\lambda)|$ is $\le M_x$

- thus $y(t) \le \int_{-\infty}^{t} M_x\, u(t+1)$

for all $t \le -1$ $y(t) = 0$ since $u(t+1)$ is 0 in this domain

$\Rightarrow y(t) \le \int_{-\infty}^{t} M_x\, d\lambda$ for $t > -1$

$y(t) \le M_x \int_{-\infty}^{t} d\lambda \Rightarrow \infty$ Thus **Not Stable**

2. (a) $y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}\, x(\tau)\, d\tau$

(i) $h_1(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}\, \delta(\tau)\, d\tau \xRightarrow{\text{Sampling Property}} \int_{-\infty}^{t} e^{-2(t-0)}\, \delta(\tau)\, d\tau$

$= e^{-2t} \int_{-\infty}^{t} \delta(\tau)\, d\tau = e^{-2t} \left[u(\tau) \Big|_{-\infty}^{t} \right] = e^{-2t} \left[u(t) - \overset{0}{\cancel{u(-\infty)}} \right]$

$= \boxed{e^{-2t}\, u(t)}$

(ii) Using $h_1(t) = e^{-2t}u(t)$

We can say $y(t) = h_1(t) * x(t)$

For $x(t) = 3e^{-4t}u(t-2)$

$$y(t) = 3e^{-4t}u(t-1) * e^{-2t}u(t)$$

$$= 3[e^{-4t}u(t-1) * e^{-2t}u(t)] = 3[e^{-2t}u(t) * e^{-4t}u(t-1)]$$

Given :

$$e^{at}u(t) * e^{bt}u(t) = \frac{1}{a-b}e^{at}u(t) + \frac{1}{b-a}e^{bt}u(t)$$

Convolution is time-invariant thus we have :

$$3[e^{-2t}u(t) * e^{-4t}u(t-1)] = \frac{1}{-2+4}e^{-2(t-1)}u(t-1) + \frac{1}{-4+2}e^{-4(t-1)}u(t-1)$$

$$= 3\left[-\frac{1}{2}e^{-2(t-1)}u(t-1) - \frac{1}{2}e^{-4(t-1)}u(t-1)\right]$$

$$= \boxed{-\frac{3}{2}u(t-1)\left[e^{-2(t-1)} + e^{-4(t-1)}\right]}$$

(iii) Using $h_1(t) = e^{-2t}u(t)$

We can say $y(t) = h_1(t) * x(t)$

For $x(t) = u(t) - u(t-3)$

$$y(t) = e^{-2t}u(t) * [u(t) - u(t-3)]$$

$$= e^{-2t}u(t) * u(t) - e^{-2t}u(t) * u(t-3)$$

$$= -\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) - \left[-\frac{1}{2}e^{-2(t-3)}u(t-3) + \frac{1}{2}u(t-3)\right]$$

$$= \boxed{-\frac{1}{2}e^{-2t}u(t) + \frac{1}{2}u(t) + \frac{1}{2}e^{-2(t-3)}u(t-3) - \frac{1}{2}u(t-3)}$$

(iv) Using $h_1(t) = e^{-2t}u(t)$

We can say $y(t) = h_1(t) * x(t)$

For $x(t) = \delta(t) - \delta(t-3)$

$$y(t) = e^{-2t}u(t) * [\delta(t) - \delta(t-3)]$$

$$= e^{-2t}u(t) * \delta(t) - e^{-2t}u(t) * \delta(t-3)$$

By sifting property

$$= \boxed{e^{-2t}u(t) - e^{-2(t-3)}u(t-3)}$$

(b) (i) We know $y(t) = x(t) * h(t)$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking the derivative with respect to t we have

$$\frac{d}{dt} y(t) = \frac{d}{dt} \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Swapping the order of integration and differentiation

$$\Rightarrow \frac{d}{dt} y(t) = \int_{-\infty}^{\infty} \frac{d}{dt}\left[x(\tau) h(t-\tau)\right] d\tau$$ $h(t-\tau)$ is the only t dependent term

$$\Rightarrow \frac{d}{dt} y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \left(\frac{d}{dt} h(t-\tau)\right) d\tau$$

The right hand side is then just $x(t) * \frac{d}{dt} h(t)$

Thus $$\frac{d}{dt} y(t) = x(t) * \left(\frac{d}{dt} h(t)\right)$$

which is what we are asked to show.

(ii) Using the same reasoning as (b)(i) along with convolution being commutative, we have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Using the same differentiating and swapping order technique we have

$$\frac{d}{dt} y(t) = \int_{-\infty}^{\infty} \frac{d}{dt}\left[h(\tau) x(t-\tau)\right] d\tau.$$ $x(t-\tau)$ is the only t dependent term

$$\frac{d}{dt} x(t) = \int_{-\infty}^{\infty} \left(\frac{d}{dt} x(t-\tau)\right) \cdot h(\tau)\, d\tau$$

$\Rightarrow$ This is simply $\left(\frac{d}{dt} x(t)\right) * h(t)$ thus we have

$$\frac{d}{dt} y(t) = \left(\frac{d}{dt} x(t)\right) * h(t)$$

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3 (a) $C_k = \frac{1}{T}\int_{-\pi}^{\pi} f(t) e^{-jk\omega_0 t}\, dt$

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-jk\omega_0 t}\, dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\cdot e^{-jk\omega_0 t}\, dt + \int_{0}^{\pi} 1\cdot e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{2\pi}\left[-\int_{-\pi}^{0} e^{-jk\omega_0 t}\, dt + \int_{0}^{\pi} e^{-jk\omega_0 t}\, dt\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{jk\omega_0}\left(e^{-jk\omega_0 t}\Big|_{-\pi}^{0}\right) - \frac{1}{jk\omega_0}\left(e^{-jk\omega_0 t}\Big|_{0}^{\pi}\right)\right]$$

With $T_0 = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{T_0} = 1$

$$= \frac{1}{2\pi}\left[\frac{1}{jk}\left(1 - e^{jk\pi}\right) - \frac{1}{jk}\left(e^{-jk\pi} - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{jk} - \frac{e^{jk\pi}}{jk} - \frac{e^{-jk\pi}}{jk} + \frac{1}{jk}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2 - e^{jk\pi} - e^{-jk\pi}}{jk}\right] = \boxed{\frac{2 - e^{jk\pi} - e^{-jk\pi}}{2jk\pi}}$$

(b) Given $x(t)$ is periodic with $C_k = \begin{cases} 2 & k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{Otherwise} \end{cases}$

(i) Is $x(t)$ real?

If $x(t)$ is real then $C_k = C^*_{-k}$.

Let $k=1$ then $C_1 = j\frac{1}{2}$ and $C_{-1} = j\frac{1}{2}$

C_{-1}^* is then $= -j\frac{1}{2}$

Since $C_1 \neq C_{-1}^*$ as $j\frac{1}{2} \neq -j\frac{1}{2}$

$x(t)$ is NOT Real

(ii) Is $x(t)$ even?

Use $x(t) = x(-t) \Rightarrow$ Then $C_k = C_{-k}$

Valid for $k=0$

For $k \neq 0$: $C_k = j\left(\frac{1}{2}\right)^{|k|}$

Then $C_{-k} = j\left(\frac{1}{2}\right)^{|-k|} = j\left(\frac{1}{2}\right)^{|k|}$ Since $C_k = C_{-k}$ the Function $x(t)$ IS Even

(iii) $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$$\frac{dx(t)}{dt} = \frac{d}{dt}\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left[c_k e^{jk\omega_0 t}\right]$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{c_k \cdot jk\omega_0}_{\hat{c}_k} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \hat{c}_k e^{jk\omega_0 t}$$

$\hat{c}_k = c_k \cdot jk\omega_0$. For $\frac{dx(t)}{dt}$ to be even $\hat{c}_k = \hat{c}_{-k}$

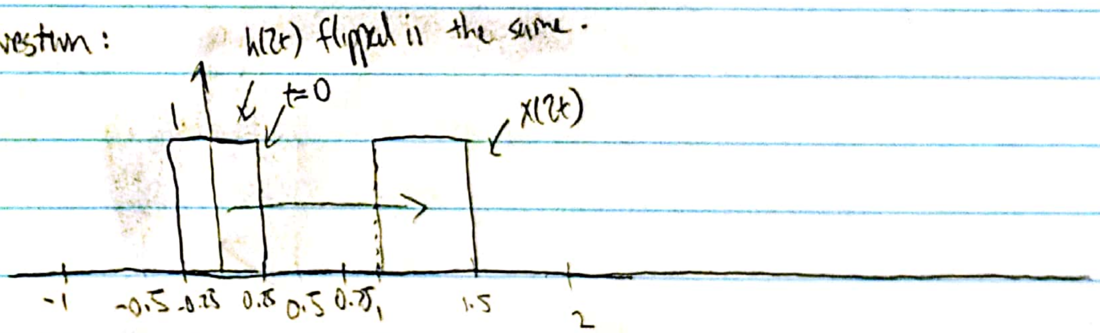
Valid for $k>0$

For $k\neq 0$: $\hat{c}_k = c_k \cdot jk\omega_0 = j\left(\frac{1}{2}\right)^{|k|} jk\omega_0 = -\left(\frac{1}{2}\right)^{|k|} k\omega_0$

$\hat{c}_{-k} = c_{-k}\, j(-k)\omega_0 = j\left(\frac{1}{2}\right)^{|-k|} \cdot j(-k)\omega_0 = \left(\frac{1}{2}\right)^{|k|} k\omega_0$

Thus $\hat{c}_k \neq \hat{c}_{-k}$, therefore $\frac{dx(t)}{dt}$ is NOT Even

Bonus Question:



For $t<0.75$ $h(2t) * x(2t) = 0$

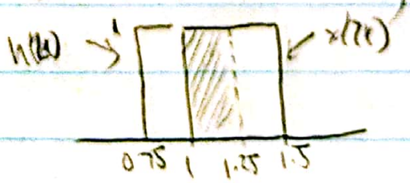
Similarly for $t>1.75$ $h(2t) * x(2t) = 0$

From 0.75 to 1.75 we have overlap reaches max value of $\frac{1}{2}\cdot 1 = \frac{1}{2}$ at $t=1.25$

Linearly goes from 0 to ½ then ½ to 0 from $t=0.75$ to 1.25 and from 1.25 to 1.75 respectively

Thus we have at **$t=0.5$ $y(0.5) = 0$**

and at $t=1$ the following



thus **$y(1) = \frac{1}{4}\cdot 1 = 1/4$**