ECE102, Spring 2020 **Midterm**

Signals & Systems Prof. A. Kadambi

University of California, Los Angeles; Department of ECE TAs: A. Safonov & G. Zhao

UCLA True Bruin academic integrity principles apply.

This exam is open book and open note, but you may not perform an internet search to seek a worked solution to a problem. Collaboration is not allowed.

8:00 am Wednesday, 29 Apr 2020
- 8:00 am Thursday, 30 Apr 2020.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Problem 1	____	/ 25
Problem 2	____	/ 22
Problem 3	____	/ 26
Problem 4	____	/ 27
BONUS	____	/ 6 bonus points
Total	____	/ 100 points + 6 bonus points

1. **Signal Properties** (25 points)

(a) (5 points) *Complex numbers.* Compute the real and imaginary parts of the complex number:

$$x(t) = e^{jt}(cos(3t) + sin(t)), \tag{1}$$

t is real.

$$x(t) = (\cos(t) + j\sin(t))(\cos(3t) + \sin(t)) = \cos(t)\cos(3t) + \cos(t)\sin(t) + j\sin(t)\cos(3t) + j\sin^2(t)$$

$$\text{real}(x(t)) = \cos(t)\cos(3t) + \cos(t)\sin(t)$$

$$\text{imaginary}(x(t)) = \sin(t)\cos(3t) + \sin^2(t)$$

(b) (5 points) *Energy and power.* What are the energy and power of this signal?

$$x(t) = Ae^{-at}u(t) \quad with \quad a > 0. \tag{2}$$

$$E_x = \lim_{T\to\infty}\int_{-T}^{T}|Ae^{-at}u(t)|^2 dt = \lim_{T\to\infty}\int_{-T}^{T}|Ae^{-at}|^2 u(t)dt = \lim_{T\to\infty}\int_{0}^{T}|A|^2 e^{-2at} = |A|^2 \cdot \lim_{T\to\infty}\left(\frac{e^{-2at}}{-2a}\Big|_0^T\right) = \frac{|A|^2}{2a} = \frac{A^2}{2a}$$

$$P_x = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|Ae^{-at}u(t)|^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T}|A|^2 e^{-2at} dt = |A|^2 \lim_{T\to\infty}\frac{1}{2T}\left(\frac{e^{-2at}}{-2a}\Big|_0^T\right) = |A|^2 \lim_{T\to\infty}\frac{1-e^{-2aT}}{4aT} = 0$$

(c) (5 points) If $x(t)$ is an even function, and $x(t-1)$ is also even, is $x(t)$ periodic? Explain your reasoning. given: $x(t) = x(-t)$ and $x(t-1) = x(-t-1)$

yes

for some constant c:

$x(c-1) = x(-c-1) = x(c+1)$

from 2nd equation   from first equation

then for $d = c+1$

$x(d+1-1) = x(d+1+1)$

$\therefore x(d) = x(d+2)$ thus x is periodic with a period of 2

(d) (5 points) Assume that the signal $x(t)$ is periodic with period T_0, and that $x(t)$ is odd (i.e. $x(t) = -x(-t)$). What is the value of $x(T_0)$? Explain your reasoning.

$x(T_0) = 0$ because for an odd function, $x(0) = 0$ since for $x(0) = -x(-0) = -x(0)$, the only number that is equal to the negative of itself is 0. Since $x(t)$ is periodic with period T_0, we know $x(T_0) = x(T_0 - T_0) = x(0) = 0$, thus $x(T_0) = 0$

(e) (5 points) Evaluate the expression

$$\int_0^\infty \delta(t-2)t^2 dt.$$

$$\int_0^\infty \delta(t-2)t^2 dt = \int_0^\infty \delta(t-2)2^2 dt = 4\int_0^\infty \delta(t-2)dt = 4$$

2. **System Properties** (22 points)

(a) (12 points) A system with input $x(t)$ and output $y(t)$ can be linear, time-invariant, causal or stable. Determine which of these properties hold for the following system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2).$$

$y(t) = S(x(t))$

linear: $S(ax_1(t)+bx_2(t)) = \frac{ax_1(t-1)+bx_2(t-1)}{t} + ax_1(t-2)+bx_2(t-2) = \frac{ax_1(t-1)}{t} + ax_1(t-2) + \frac{bx_2(t-1)}{t} + bx_2(t-2)$
$= aS(x_1(t)) + bS(x_2(t))$

not time-invariant:

shift input→output: $S(x(t-s)) = \frac{x(t-s-1)}{t} + x(t-s-2)$

shift output: $y(t-s) = \frac{x(t-s-1)}{t-s} + x(t-s-2)$ not the same, not time-invariant

causal: $y(t)$ relies on $x(t-1)$ and $x(t-2)$, thus it only relies on values of $x(t)$ in the past, thus it is causal

not stable: $y(0) = \frac{x(-1)}{0} + x(-2)$ is unbounded even with a bounded $x(t)$, thus the system is not stable

(b) (10 points) Determine if each of the following statements concerning LTI systems is true or false. Explain your reasoning.

i. If h(t) is the impulse response of an LTI system and h(t) is periodic and nonzero, the system is unstable.

true

$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty$ if $h(t)$ is periodic and nonzero, thus the system producing the impulse response $h(t)$ is unstable

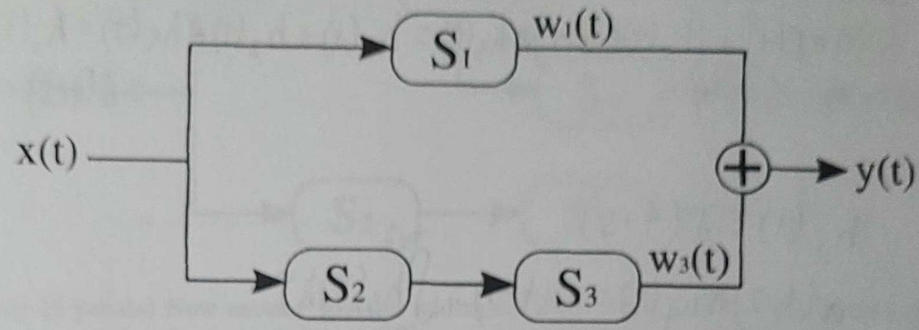
since the magnitude is nonzero, integrating over a single period would yield a positive value, so integrating over infinite periods from $-\infty$ to ∞ would yield an infinite value

ii. If an LTI system is causal, it is stable.

false

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ is causal since it only relies on past and present values of $x(t)$, but it is not stable because plugging in a bounded input $x(t)=1$ results in $y(t) = \int_{-\infty}^{t} 1\,dt = t+\infty$ which is unbounded

3. **System Response of LTI system** (26 points)



It is given that Systems 1, 2 and 3 are all LTI systems, which are used in parts a through d. Note that $w_1(t)$ and $w_3(t)$ are the outputs of Systems 1 and 3, respectively. Let $h_1(t)$, $h_2(t)$ and $h_3(t)$ represent the impulse response for System 1, 2 and 3, respectively.
For parts (a) through (d), we have prior knowledge of Subsystem 2

$$h_2(t) = u(t-2).$$

For parts (a) through (c), we also have prior knowledge of the input/output mapping of the entire system

$$y(t) = x(t+5) + \int_{-\infty}^{t} x(\tau-3)d\tau.$$

(a) (6 points) What is the impulse response of the entire system (i.e. S_{eq})? What is the step response of S_{eq}?

$h_{eq} = \delta(t+5) + \int_{-\infty}^{t} \delta(\tau-3)d\tau = \delta(t+5) + u(t-3)$

impulse response

$S_{eq} = u(t+5) + \int_{-\infty}^{t} u(\tau-3)d\tau = u(t+5) + r(t-3)$

step response

(b) (8 points) Find $h_1(t)$ and $h_3(t)$ that satisfies the input and output relationship that is given. It might be useful to determine the values of $w_1(t)$ and $w_3(t)$ first.

$$h_{eq}(t) = h_1(t) * \delta(t) + (h_2(t) * \delta(t)) * h_3(t) = h_1(t) + h_2(t) * h_3(t) = h_1(t) + u(t-2) * h_3(t)$$
$$= \delta(t+5) + u(t-3)$$

$$h_1(t) = \delta(t+5)$$

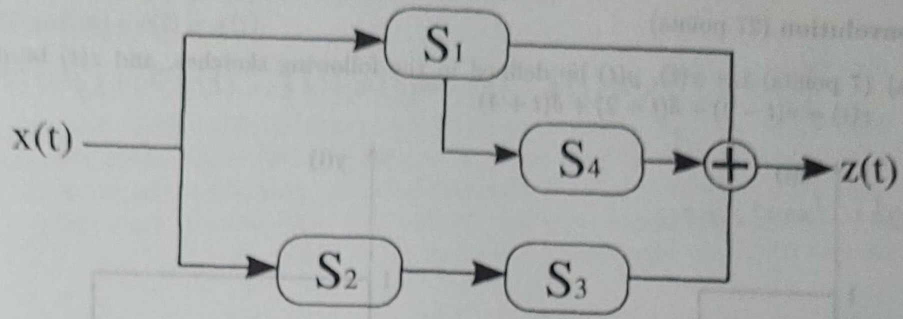
$$u(t-2) * h_3(t) = u(t-3) = \int_{-\infty}^{t-2} h_3(\tau)\,d\tau$$

$$\delta(t-3) = h_3(t-2)$$

$$h_3(t) = \delta(t-1)$$

(c) (6 points) Using the solution from parts a and b, is System 1 Causal/Stable? Is the other subsystem (Cascaded Systems 2 and 3) Causal/Stable? Finally, is S_{eq} Causal/Stable?

- System 1 is <u>not causal</u> since its impulse response is nonzero at $t=-5$: $h_1(-5) = \delta(-5+5) = \delta(0)$, thus since the impulse response is nonzero at a time $t<0$, the system is not causal.
- System 1 is <u>stable</u> since $\int_{-\infty}^{\infty} |h_1(t)|\,dt = \int_{-\infty}^{\infty} \delta(t+5)\,dt = 1 < \infty$
- Cascaded systems 2 and 3 is ~~[illegible]~~ <u>causal</u> since $(h_3 * h_2)(t) = u(t-3)$ is 0 for times $t<0$
- Cascaded systems 2 and 3 is <u>not stable</u> since $\int_{-\infty}^{\infty} |(h_3 * h_2)(t)|\,dt = \int_{-\infty}^{\infty} u(t-3)\,dt = \infty$ which is not finite, thus the system is not stable
- S_{eq} is <u>not causal</u> since its impulse response is nonzero at $t=-5$: $h_{eq}(-5) = \delta(-5+5) + u(-5-3) = \delta(0)$, thus since the impulse response is nonzero at ~~[illegible]~~ a time $t<0$, the system is not causal
- S_{eq} is <u>not stable</u> since $\int_{-\infty}^{\infty} |h_{eq}(t)|\,dt = \int_{-\infty}^{\infty} (\delta(t+5) + u(t-3)) = \infty$ which is not finite, thus the system is not stable



(d) (6 points) Now assume that an additional LTI subsystem is added as seen above. Assume that Systems 1, 2 and 3 remain the same, with the input/output mapping being changed due to the introduction of System 4. What is the impulse response of the new equivalent system? Give an expression that is in terms of $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$. Next, give a specific equation for $h_4(t)$ that will allow for the new equivalent system (S^*_{eq}) to be Stable.

$$h^*_{eq} = h_1 + h_1 * h_4 + h_2 * h_3$$

$$h^*_{eq} = \delta(t+5) + \delta(t+5) * h_4 + u(t-3)$$

assume $h_4 = -u(t-8)$, then $\delta(t+5) * h_4 = (-u(t-8)) * \delta(t+5)$

$$= \int_{-\infty}^{\infty} -u(\tau-8)\delta(t-\tau+5)d\tau$$

$$= -\int_{-\infty}^{\infty} u(t-3)\delta(t-\tau+5)d\tau = u(t-3)$$

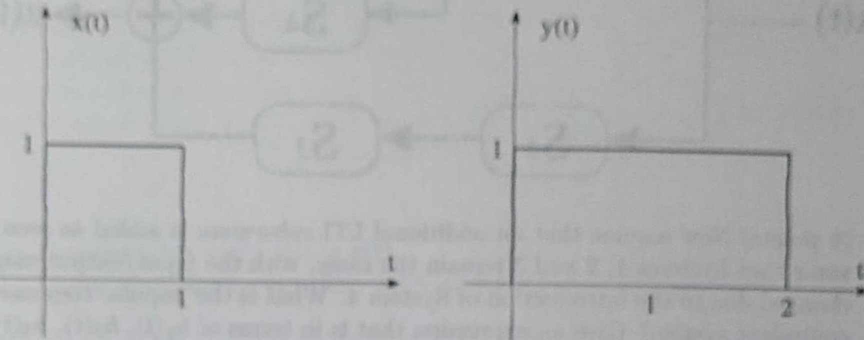
then $h^*_{eq} = \delta(t+5) - u(t-3) + u(t-3) = \delta(t+5)$

thus h^*_{eq} is the impulse response of a stable system since

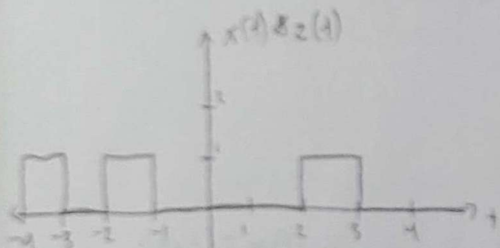
$$\int_{-\infty}^{\infty} |h^*_{eq}| dt = \int_{-\infty}^{\infty} \delta(t+5) dt = 1 < \infty$$

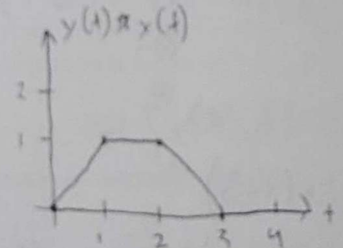
4. **Convolution** (27 points)

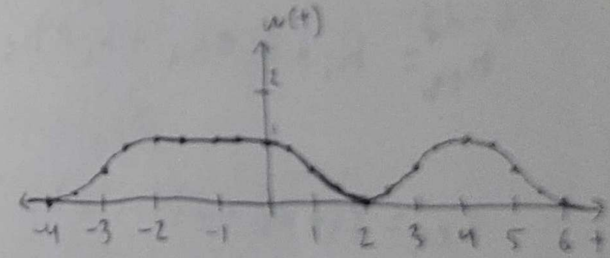
(a) (7 points) Let $x(t)$, $y(t)$ be defined in the following sketches, and $z(t)$ be defined as $z(t) = \delta(t-2) + \delta(t+2) + \delta(t+4)$.



Plot w(t), which is given by $w(t) = x(t) * z(t) * y(t) * x(t)$. Show work (e.g. plot intermediate results).







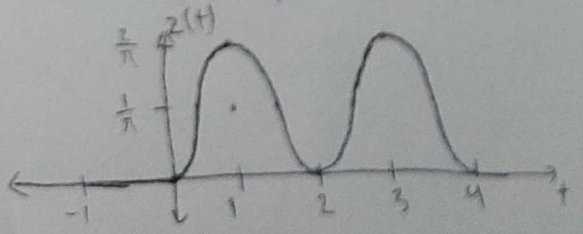
(b) (8 points) Let $x(t) = sin(\pi t)u(t)$ and $y(t) = u(t) - u(t-4)$. Compute and plot the output $z(t) = x(t) * y(t)$.

$$z(t) = \sin(\pi t)u(t) * (u(t) - u(t-4)) = \sin(\pi t)u(t) * u(t) - \sin(\pi t)u(t) * u(t-4)$$

$$= \int_{-\infty}^{\infty} \sin(\pi \tau)u(\tau)u(t-\tau)\,d\tau - \int_{-\infty}^{\infty} \sin(\pi\tau)u(\tau)u(t-\tau-4)\,d\tau = \int_{-\infty}^{t} \sin(\pi\tau)u(\tau)\,d\tau - \int_{-\infty}^{t-4} \sin(\pi\tau)u(\tau)\,d\tau$$

$$= \left(\begin{cases} \int_0^t \sin(\pi\tau)\,d\tau, & t \geq 0 \\ 0, & t < 0 \end{cases}\right) - \left(\begin{cases} \int_0^{t-4} \sin(\pi\tau)\,d\tau, & t \geq 4 \\ 0, & t < 4 \end{cases}\right) = \left(\begin{cases} \frac{1-\cos(\pi t)}{\pi}, & t \geq 0 \\ 0, & t < 0 \end{cases}\right) - \left(\begin{cases} \frac{1-\cos(\pi t - 4\pi)}{\pi}, & t \geq 4 \\ 0, & t < 4 \end{cases}\right)$$

$$= \left(\begin{cases} \frac{1-\cos(\pi t)}{\pi}, & t \geq 0 \\ 0, & t < 0 \end{cases}\right) - \left(\begin{cases} \frac{1-\cos(\pi t)}{\pi}, & t \geq 4 \\ 0, & t < 4 \end{cases}\right) = \begin{cases} \frac{1-\cos(\pi t)}{\pi}, & 0 \leq t \leq 4 \\ 0, & t < 0 \text{ or } t > 4 \end{cases}$$



(c) (6 points) Prove or disprove the following equality:
$\delta(t)\cos(2t) * x(t) = x(t)$.

$\delta(t)\cos(2t) * x(t) = \delta(t)\cos(0) * x(t) = \delta(t) * x(t) = x(t)$

convolutional identity

(d) (6 points) Let $x(t) * y(t) = z(t)$.
Verify the following equality: $x(t-a) * y(t-2a) = z(t-3a)$

left side:

$x(t-a) * y(t-2a) = \int_{-\infty}^{\infty} x(\psi - a)\, y(t-\psi-2a)\, d\psi$

substitute: $s = \psi - a$
$ds = d\psi$

$= \int_{-\infty}^{\infty} x(s)\, y(t-(s+a)-2a)\, ds$

$= \int_{-\infty}^{\infty} x(s)\, y(t-s-3a)\, ds$

they are the same

right side:

$z(t-3a) = \int_{-\infty}^{\infty} x(\psi)\, y(t-\psi-3a)\, d\psi$

5. Bonus (6 points)

(This question was used in an industry interview to use system modeling of a real-world problem.)

Self-driving cars have devices on top of the car known as LiDAR (light detection and ranging). LiDAR uses the *time of flight* principle to obtain the 3D shape of the surrounding environment. LiDAR works as follows: a packet of photons gets sent to the scene - it bounces off an object a range of z meters away and returns to the LiDAR unit. Since we know how fast the photons travel, we can compute the range of the object.

First, recall that the round-trip distance is

$$distance = velocity \times t_d. \tag{3}$$

where distance is the round-trip distance of travel (from the source, to the object and back) and t_d is the time the trip takes to occur. We are interested in the range of the object which is

$$range = distance \times 0.5 = velocity \times t_d \times 0.5. \tag{4}$$

In the specific case of sending a laser pulse to the target, we know the speed of photons is $c = 3 \times 10^8$ meters per second. Therefore, the range is computed as:

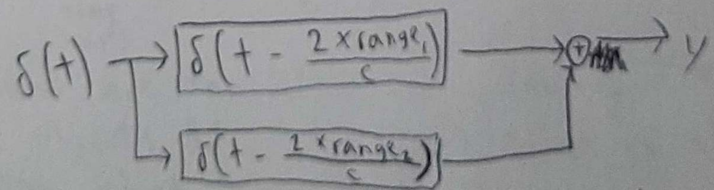
$$range = c \times t_d \times 0.5, \tag{5}$$

and we only need to measure the time delay of the pulse to obtain the range. Using the ideas from this class, this can be written as a system.

$$\delta(t) \rightarrow \boxed{\delta\left(t - \frac{2 \times range}{c}\right)} \rightarrow y, \tag{6}$$

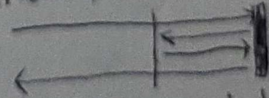
Here the laser pulse that we sent is $\delta(t)$, and it goes into a system that introduces a delay to yield the measurement. This is an idealized model of LiDAR that does not involve any multipath and assumes the laser can emit a Dirac pulse.

(a) (2 points) Please draw the equivalent system diagram if there is a transparent window at $range_1$ and a brick wall behind it at $range_2$. We will assume that the impulse response of the transparent window and brick wall can be modeled as Dirac Delta functions.



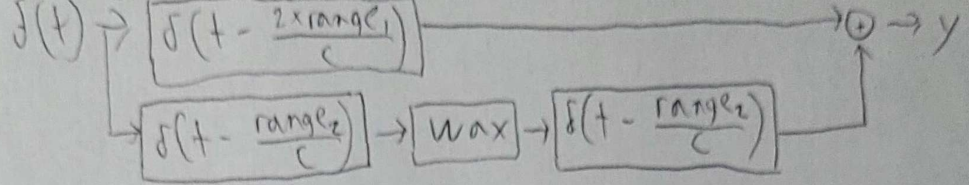
*assuming the light will bounce off the glass back to the sensor, but will also pass through and bounce off the wall back through the glass and to the sensor

if you also consider light reflecting multiple times like shown:



then also include systems $\boxed{\delta\left(t - \frac{2 \times (range_n)}{c}\right)}$ in parallel for n from 3 to ∞ with $range_n = range_2 + (n-2)(range_2 - range_1)$

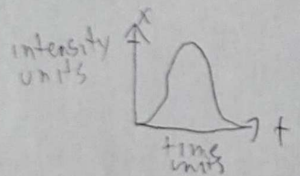
(b) (2 points) Please draw the equivalent system diagram if the brick wall in part (a) is changed to a block of wax. Please justify and/or explain your answer.



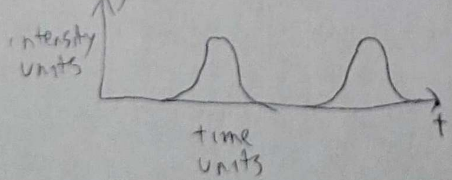
Since we do not know the impulse response of the wax I put it between two delays of the light getting to the wax and coming back. The wax system is the system that takes in light from a range of 0 and outputs what is reflected.

(c) (2 points) Now we will assume that the laser is a cheap laser. It cannot fire a short pulse that mimics a Dirac. In fact, it's more like the lightbulb in your room, which takes a finite time to turn on. Using your understanding of signals and systems, explain concisely how the output y will be changed and why this could be a problem in the multipath case.

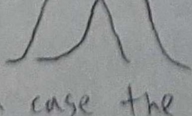
Assume the light turning on and off is more like this:



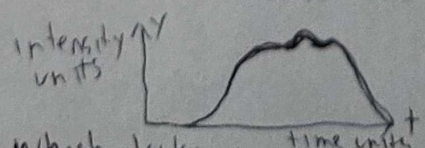
If the glass and walls have impulse responses of shifted diracs, then if the light flash is very fast we might get a y that looks like this



Clearly there are two pulses here, indicating the light bounced of two objects. But if the light flash is slower, the two pulses may start to overlap:



In which case the y might look like this:



which looks like one single pulse, even though it is the sum of two received pulses. So our sensor would detect 1 object instead of 2.