ECE102, Spring 2020 | **Midterm**
Signals & Systems | Prof. A. Kadambi
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UCLA True Bruin academic integrity principles apply.

This exam is open book and open note, but you may not perform an internet search to seek a worked solution to a problem. Collaboration is not allowed.

8:00 am Wednesday, 29 Apr 2020
- 8:00 am Thursday, 30 Apr 2020.

State your assumptions and reasoning.
No credit without reasoning.
Show all work on these pages.

Problem 1	____ / 25
Problem 2	____ / 22
Problem 3	____ / 26
Problem 4	____ / 27
BONUS	____ / 6 bonus points
Total	____ / 100 points + 6 bonus points

1. **Signal Properties** (25 points)

(a) (5 points) *Complex numbers.* Compute the real and imaginary parts of the complex number:

$$x(t) = e^{jt}(cos(3t) + sin(t)), \quad (1)$$

t is real.

$X(t) = (\cos(t) + j\sin(t))(\cos(3t) + \sin(t))$

$= \cos(t)\cos(3t) + \cos(t)\sin(t) + j\sin(t)\cos(3t) + j\sin^2(t)$

$= \cos(t)\cos(3t) + \cos(t)\sin(t) + j(\sin(t)\cos(3t) + \sin^2(t))$

$\boxed{Re(x(t)) = \cos(t)\cos(3t) + \cos(t)\sin(t)}$ $\boxed{Im(x(t)) = \sin(t)\cos(3t) + \sin^2(t)}$

(b) (5 points) *Energy and power.* What are the energy and power of this signal?

$$x(t) = Ae^{-at}u(t) \quad with \quad a > 0. \quad (2)$$

$|x(t)|^2 = Ae^{-at}u(t) \cdot Ae^{-at}u(t)$

$= A^2 e^{-2at} u(t)$

$E = \int_{-\infty}^{\infty} A^2 e^{-2at} u(t)\, dt = \int_0^{\infty} A^2 e^{-2at} dt$

$= A^2 \frac{e^{-2at}}{-2a}\Big|_0^{\infty} = A^2 \frac{1}{-2ae^{2at}}\Big|_0^{\infty} = A^2 \cdot \frac{1}{-\infty} - \left(A^2 \cdot \frac{1}{-2a}\right) = 0 + \frac{A^2}{2a} \cdot \frac{A^2}{2a}$

$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \cdot \frac{A^2}{2a}$

$= \frac{1}{\infty} \cdot \frac{A^2}{2a} = 0$

$\boxed{E = \frac{A^2}{2a}}$

$\boxed{P = 0}$

(c) (5 points) If $x(t)$ is an even function, and $x(t-1)$ is also even, is $x(t)$ periodic? Explain your reasoning.

$X(t) = x(-t)$

$x(t') = x(-t')$

for $t' = t-1$

$x(t-1) = x(-t+1)$

if periodic $X(t+T_0) = x(t) \ \forall t$

$X(1) = X(-1)$

$X(1) = X(-3)$

$X(t) = X(-t) = X(t-2)$ for all t

$X(t)$ is periodic. By the given statements, we know $X(t) = X(-t) = X(t-2)$ for all t. This is the only way to satisfy both periodic statements. Therefore for every t, $x(t)$, equals the value 2 units away from it, making $x(t)$ periodic

(d) (5 points) Assume that the signal $x(t)$ is periodic with period T_0, and that $x(t)$ is odd (i.e. $x(t) = -x(-t)$). What is the value of $x(T_0)$? Explain your reasoning.

$x(t) = x(t+T_0)$

$x(t) = -x(-t)$

$x(-t) = x(-t+T_0)$

$X(t-T_0) = X(t)$

plug in T_0

$X(T_0 - T_0) = X(T_0)$

$X(0) = X(T_0)$

$\boxed{x(T_0) = X(0)}$

If $x(t)$ has period T_0, then $x(t)$ has the same value at $t+T_0$ and $t-T_0$, $x(t) = x(t-T_0)$. If you plug in T_0, you get $x(T_0) = x(0)$.

(e) (5 points) Evaluate the expression

$$\int_0^\infty \delta(t-2)t^2dt.$$

$= \int_0^\infty \delta(t-2)(2)^2 dt$

$= 4\int_0^\infty \delta(t-2)\,dt$

$= \boxed{4}$

$f(t) = t^2$

$\int_0^\infty \delta(t-2) f(t)\,dt$

$\int_0^\infty \delta(t-2) f(2)\,dt$

$f(2)\int_0^\infty \delta(t-2)\,dt$

$= f(2)\cdot 1$

$= 4$

2. **System Properties** (22 points)

(a) (12 points) A system with input $x(t)$ and output $y(t)$ can be linear, time-invariant, causal or stable. Determine which of these properties hold for the following system. Explain your answer.

$$y(t) = \frac{x(t-1)}{t} + x(t-2).$$

at $t=0$ $\quad y(0) = \frac{x(-1)}{0} + x(-2) = \infty$

$\boxed{\text{not stable}}$

$\boxed{\text{Causal}}$ $x(t-1)$ and $x(t-2)$ never depend on future values for any t value

delayed input $\quad \frac{x(t-\tau-1)}{t} + x(t-\tau-2)$

$\neq$

delayed output $\quad y(t-\tau) = \frac{x(t-\tau-1)}{t-\tau} + x(t-\tau-2)$

$\boxed{\text{Not time-invariant}}$ delayed input $\neq$ delayed output

$S(ax(t)+b\tilde{x}(t)) = aS(x)+bS(\tilde{x})$

$S(ax+b\tilde{x}) = \frac{ax(t-1)}{t} + ax(t-2) + \frac{b\tilde{x}(t-1)}{t} + b\tilde{x}(t-2)$

$= a\left(\frac{x(t-1)}{t} + x(t-2)\right) + b\left(\frac{\tilde{x}(t-1)}{t} + \tilde{x}(t-2)\right)$

$= aS(x) + bS(\tilde{x})$

$\boxed{\text{Linear}}$

(b) (10 points) Determine if each of the following statements concerning LTI systems is true or false. Explain your reasoning.

i. If h(t) is the impulse response of an LTI system and h(t) is periodic and nonzero, the system is unstable.

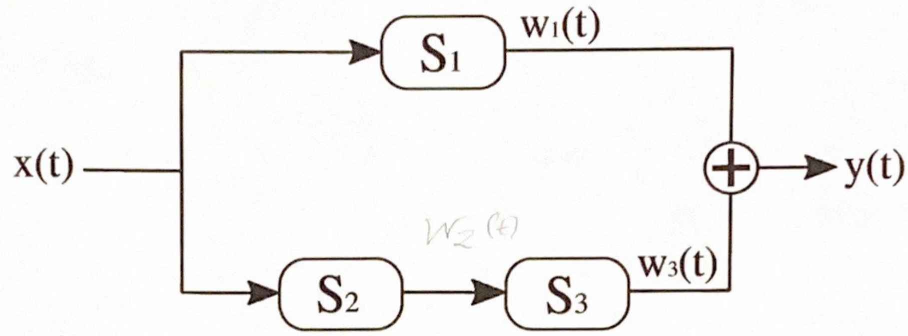
True. For a periodic impulse response, there will always be an input function that approaches infinity as $t \to \pm\infty$ that will cause the convolution integral to evaluate to an infinite quantity. The periodicity of the impulse function makes it impossible to limit the bounds of the integral, so even if you have an input function that is bounded for all t, the integral will evaluate to infinity for some input.

ii. If an LTI system is causal, it is stable.

False $S(x(t)) = \frac{1}{1 - x(t)}$

if $x(t) = 1$, $S(x(t)) = \infty$ regardless of its causality.

3. **System Response of LTI system** (26 points)



It is given that Systems 1, 2 and 3 are all LTI systems, which are used in parts a through d. Note that $w_1(t)$ and $w_3(t)$ are the outputs of Systems 1 and 3, respectively. Let $h_1(t)$, $h_2(t)$ and $h_3(t)$ represent the impulse response for System 1, 2 and 3, respectively.
For parts (a) through (d), we have prior knowledge of Subsystem 2

$$h_2(t) = u(t-2).$$

For parts (a) through (c), we also have prior knowledge of the input/output mapping of the entire system

$$y(t) = x(t+5) + \int_{-\infty}^{t} x(\tau - 3)d\tau.$$

(a) (6 points) What is the impulse response of the entire system (i.e. S_{eq})? What is the step response of S_{eq}?

$y(t) = x(t+5) + \int_{-\infty}^{\infty} x(\tau-3)u(t-\tau)d\tau$

$= x(t+5) + x(t-3) * u(t)$

if $x(t) = \delta(t)$, $y(t) = h(t)$

$h(t) = \delta(t+5) + \int_{-\infty}^{t} \delta(\tau-3)\, d\tau$

$\boxed{h_{eq}(t) = \delta(t+5) + u(t-3)}$

$W_3(t) =$

$h_{eq}(t) = \frac{d}{dt}$ step response

$H(u(t)) = \int_{-\infty}^{\infty} \delta(t+5) + u(t-3)\, dt$

$\boxed{\text{step response} = H_{eq}(u(t)) = u(t+5) + r(t-3)}$

(b) (8 points) Find $h_1(t)$ and $h_3(t)$ that satisfies the input and output relationship that is given. It might be useful to determine the values of $w_1(t)$ and $w_3(t)$ first.

$y(t) = x(t+5) + \int_{-\infty}^{t} x(\tau-3)\,d\tau$

Assuming $w_1(t) = x(t+5)$

$w_2(t) = \int_{-\infty}^{t} x(\tau-3)\,d\tau$

$W_1(t)$: $x(t+5) = x(t) * h_1(t) = \int_{-\infty}^{\infty} x(\tau)\, h_1(t-\tau)$ let $h_1 = \delta(t+5)$

$= \int_{-\infty}^{\infty} x(\tau)\,\delta(t+5-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t+5)\,\delta(t+5-\tau)\,d\tau = x(t+5)$

$\boxed{h_1(t) = \delta(t+5)}$

Note: $x(t-3) = x(t) * \delta(t-3)$

$W_2(t)$: $\int_{-\infty}^{t} x(\tau-3)\,d\tau = \int_{-\infty}^{\infty} x(\tau-3)\,u(t-\tau)\,d\tau = x(t-3) * u(t)$

$= x(t) * \delta(t-3) * u(t) = x(t) * u(t-2) * \delta(t-1)$

Side: $\delta(t-3) * u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-3-\tau) = u(t-3)$

$\delta(t-1) * u(t-2) = \int_{-\infty}^{\infty} u(\tau-2)\,\delta(t-1-\tau)\,d\tau = u(t-1-2) = u(t-3)$

$\delta(t-3) * u(t) = \delta(t-1) * u(t-2)$

$w_2(t) = x(t) * h_2(t) * h_3(t) = x(t) * u(t-2) * \delta(t-1)$

$\boxed{h_3(t) = \delta(t-1)}$

(c) (6 points) Using the solution from parts a and b, is System 1 Causal/Stable? Is the other subsystem (Cascaded Systems 2 and 3) Causal/Stable? Finally, is S_{eq} Causal/Stable?

$S_1(x(t)) = x(t) * \delta(t+5) = x(t+5)$

Stable, when $x(t+5)$ is bounded, output is bounded

Not Causal, depends on future times

Cascaded Systems 2 and 3 $= \int_{-\infty}^{t} x(\tau-3)\,d\tau$

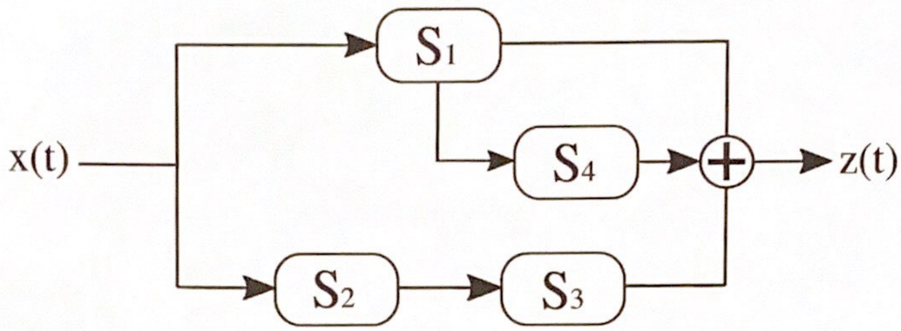
Not stable, if $x(\tau-3) = e^{-\tau}$, although the input is bounded, the output is infinite

Causal, only depends on past inputs of t

Seq

Not stable

Not causal



(d) (6 points) Now assume that an additional LTI subsystem is added as seen above. Assume that Systems 1, 2 and 3 remain the same, with the input/output mapping being changed due to the introduction of System 4. What is the impulse response of the new equivalent system? Give an expression that is in terms of $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$. Next, give a specific equation for $h_4(t)$ that will allow for the new equivalent system (S^*_{eq}) to be Stable.

$h_{eq} = \delta(t) * h_1(t) + \delta(t) * h_1(t) * h_4(t) + \delta(t) * h_2(t) * h_3(t)$

$$\boxed{h_{eq}(t) = h_1(t) + h_1(t) * h_4(t) + h_2(t) * h_3(t)}$$

$h_1(t) * h_4(t) = -h_2(t) * h_3(t)$

LHS

$\delta(t+5) * h_4(t)$ $= -u(t-2) * \delta(t-1)$

$= \int_{-\infty}^{\infty} -u(\tau-2)\,\delta(t-1-\tau)\,d\tau$

$= -u(t-1-2) = -u(t-3)$

$u(t-3) = u(t+5-8)$

if

$h_4(t) = -u(t-8)$

then

$\delta(t+5) * h_4(t) = \int_{-\infty}^{\infty} -u(\tau-8)\,\delta(t+5-\tau)\,d\tau = -u(t+5-8) = -u(t-3)$

$$\boxed{h_4(t) = -u(t-8)}$$

makes $h_{eq}(t) = h_1(t) = \delta(t+5)$

$y(t) = x(t) * \delta(t+5) = x(t+5)$

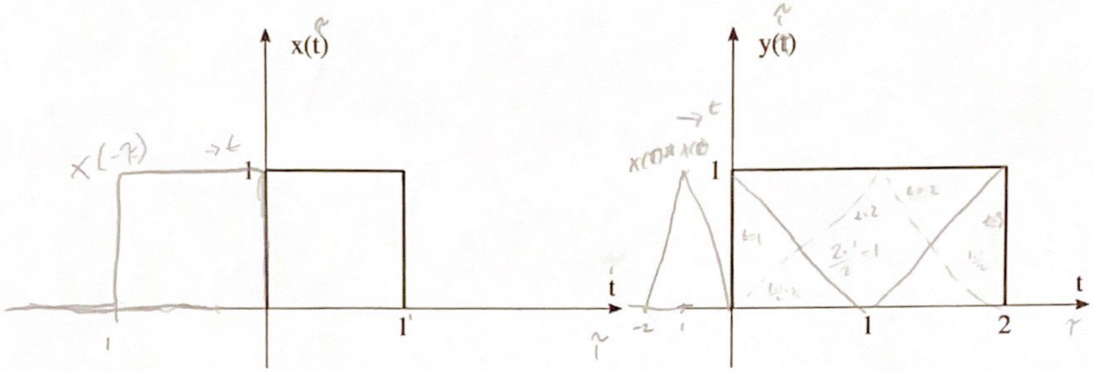
stable

not causal

$\delta(x) = rect(x) * rect(x)$

4. **Convolution** (27 points)

(a) (7 points) Let $x(t)$, $y(t)$ be defined in the following sketches, and $z(t)$ be defined as $z(t) = \delta(t-2) + \delta(t+2) + \delta(t+4)$.



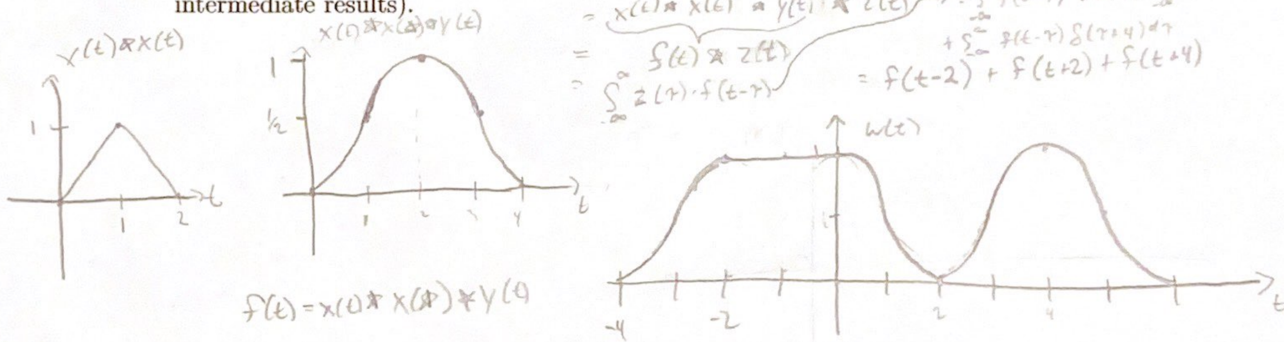
Plot w(t), which is given by $w(t) = x(t) * z(t) * y(t) * x(t)$. Show work (e.g. plot intermediate results).

$= \underbrace{x(t) * x(t) * y(t)}_{f(t)} * z(t) \to = \int_{-\infty}^{\infty} f(t-\tau)\,\delta(\tau-2)\,d\tau + \int_{-\infty}^{\infty} f(t-\tau)\,\delta(\tau+2)\,d\tau + \int_{-\infty}^{\infty} f(t-\tau)\,\delta(\tau+4)\,d\tau$

$= f(t) * z(t)$

$= \int_{-\infty}^{\infty} z(\tau) \cdot f(t-\tau)$

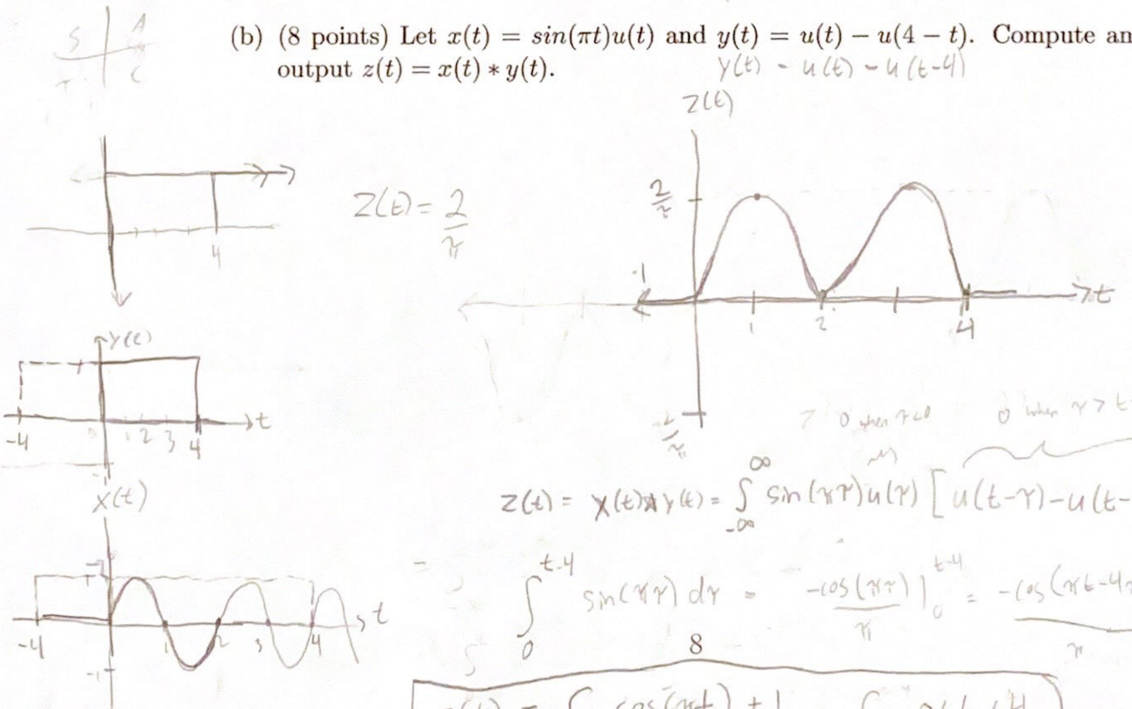
$= f(t-2) + f(t+2) + f(t+4)$



$f(t) = x(t) * x(t) * y(t)$

(b) (8 points) Let $x(t) = sin(\pi t)u(t)$ and $y(t) = u(t) - u(4-t)$. Compute and plot the output $z(t) = x(t) * y(t)$.

$y(t) = u(t) - u(t-4)$



$z(t) = \frac{2}{\pi}$

$\int_0^1 \sin(\pi t)\,d\tau$

$\frac{-\cos(\pi t)}{\pi}\Big|_0^1$

$\frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi}$

$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} \sin(\pi\tau)u(\tau)\left[u(t-\tau) - u(t-4-\tau)\right]d\tau$

$= \int_0^{t-4} \sin(\pi\tau)\,d\tau = \frac{-\cos(\pi\tau)}{\pi}\Big|_0^{t-4} = \frac{-\cos(\pi t - 4\pi)}{\pi} - \left(-\frac{\cos(0)}{\pi}\right)$

$$\boxed{z(t) = \begin{cases} \frac{-\cos(\pi t) + 1}{\pi} & \text{for } 0 < t < 4 \\ 0 & \text{otherwise} \end{cases}}$$

(c) (6 points) Prove or disprove the following equality:
$\delta(t)cos(2t) * x(t) = x(t)$.

$$\delta(t)\cos(2t) * x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\cos(2(t-\tau))\cdot x(\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(t-\tau)\cos(2(t-t))\,x(t)\,d\tau$$

$$= \int_{-\infty}^{\infty} \delta(t-\tau)\cos(0)\,x(t)\,d\tau = \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau = x(t)$$

$\boxed{\text{True}}$

(d) (6 points) Let $x(t) * y(t) = z(t)$. LHS
Verify the following equality: $x(t-a) * y(t-2a) = z(t-3a)$

LHS

$$x(t-a) * y(t-2a)$$

$$= x(t) * \delta(t-a) * y(t) * \delta(t-2a)$$

$$= x(t) * y(t) * \delta(t-a) * \delta(t-2a)$$

$$\int_{-\infty}^{\infty} \delta(\tau-a)\,\delta(t-2a-\tau)\,d\tau = \delta(t-2a-a) = \delta(t-3a)$$

$$= x(t) * y(t) * \delta(t-3a)$$

$$= z(t) * \delta(t-3a) = \int_{-\infty}^{\infty} z(\tau)\,\delta(t-3a-\tau)\,d\tau$$

$$= z(t-3a) = \text{RHS}$$

5. Bonus (6 points)

(This question was used in an industry interview to use system modeling of a real-world problem.)

Self-driving cars have devices on top of the car known as LiDAR (light detection and ranging). LiDAR uses the *time of flight* principle to obtain the 3D shape of the surrounding environment. LiDAR works as follows: a packet of photons gets sent to the scene - it bounces off an object a range of z meters away and returns to the LiDAR unit. Since we know how fast the photons travel, we can compute the range of the object.

First, recall that the round-trip distance is

$$distance = velocity \times t_d. \tag{3}$$

where distance is the round-trip distance of travel (from the source, to the object and back) and t_d is the time the trip takes to occur. We are interested in the range of the object which is

$$range = distance \times 0.5 = velocity \times t_d \times 0.5. \tag{4}$$

In the specific case of sending a laser pulse to the target, we know the speed of photons is $c = 3 \times 10^8$ meters per second. Therefore, the range is computed as:

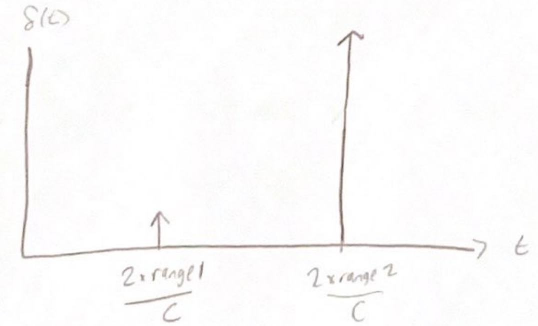
$$range = c \times t_d \times 0.5, \tag{5}$$

and we only need to measure the time delay of the pulse to obtain the range. Using the ideas from this class, this can be written as a system.

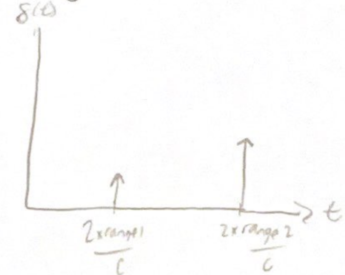
$$\delta(t) \rightarrow \boxed{\delta\left(t - \frac{2 \times range}{c}\right)} \rightarrow y, \tag{6}$$

Here the laser pulse that we sent is $\delta(t)$, and it goes into a system that introduces a delay to yield the measurement. This is an idealized model of LiDAR that does not involve any multipath and assumes the laser can emit a Dirac pulse.

(a) (2 points) Please draw the equivalent system diagram if there is a transparent window at $range_1$ and a brick wall behind it at $range_2$. We will assume that the impulse response of the transparent window and brick wall can be modeled as Dirac Delta functions.

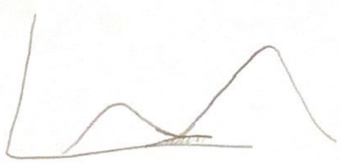


(b) (2 points) Please draw the equivalent system diagram if the brick wall in part (a) is changed to a block of wax. Please justify and/or explain your answer.



Less photons will be reflected by the wax, making its impulse response smaller.

(c) (2 points) Now we will assume that the laser is a cheap laser. It cannot fire a short pulse that mimics a Dirac. In fact, it's more like the lightbulb in your room, which takes a finite time to turn on. Using your understanding of signals and systems, explain concisely how the output y will be changed and why this could be a problem in the multipath case.



The output will no longer be a delayed dirac function, but a drawn out-smeared impulse response. In the multipath case, the impulse responses of various ranges will overlap and superimpose, making the LiDAR system detect inaccurate ranges.