Total: 60 points

Signals and Systems

Final Exam Solutions

December 11, 2013

Problem 1 (10 points) *Three independent questions on convolution.*

(a) (4 points) Clearly draw the signal $y(t) = x_1(t) \star x_2(t)$, where

$$x_1(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_2(t) = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 3 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$$

(b) (3 points) Convolution with the unit step function

$$u(t) = \begin{cases} 1\,, & t > 0\,, \\ 0\,, & t \le 0\,. \end{cases}$$

can be used as an integrator. In particular, prove that if $f(t)$ is a causal signal, then

$$\int_0^t f(y)\, dy = (u \star f)(t)\,.$$

(c) (3 points) If $h = f \star g$ show that

$$\int_{-\infty}^{\infty} h(x)\, dx = \left(\int_{-\infty}^{\infty} f(x)\, dx \right) \left(\int_{-\infty}^{\infty} g(x)\, dx \right).$$

Informally, areas multiply under convolution.

Solution:

(a) Let

$$s_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$r_1(t) = x_1(t) \star s_1(t) = \begin{cases} \int_0^t (t-\tau)\, d\tau = \frac{t^2}{2} & 0 \le t < 1 \\ \int_{t-1}^1 (t-\tau)\, d\tau = t - \frac{t^2}{2} & 1 \le t < 2 \end{cases}$$

Similarly, let

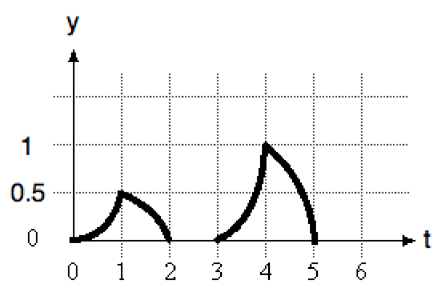
$$s_2(t) = \begin{cases} 2 & 3 \le t < 4 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$r_2(t) = x_1(t) \star s_2(t) = 2[r_1(t-3)]$$

Therefore,

$$y(t) = r_1(t) + 2[r_1(t-3)]$$



(b) Since $f(t)$ is causal, $f(t) = f(t)u(t)$. Using the definition of convolution

$$\begin{aligned} (u \star f)(t) &= \int_{-\infty}^{\infty} f(y)\, u(y)\, u(t-y)\, dy \\ &= \int_0^{\infty} f(y)\, u(t-y)\, dy \\ &= \int_0^t f(y)\, dy \end{aligned}$$

(c) Take the Fourier transform of $h = f \star g$ to get $H(\omega) = F(\omega)G(\omega)$ and evaluate at $\omega = 0$:

$$\int_{-\infty}^{\infty} h(x)\, dx = H(0) = F(0)\, G(0) = \left(\int_{-\infty}^{\infty} f(x)\, dx\right)\left(\int_{-\infty}^{\infty} g(x)\, dx\right).$$

Problem 2 (10 points)

Let $f(t)$ be a periodic signal of period 1. One says that $f(t)$ has *half-wave symmetry* if

$$f(t-\frac{1}{2}) = -f(t)\,.$$

(a) Does $f(t) = \sin(2\pi t) + \sin(4\pi t)$ have half-wave symmetry? Find its Fourier series.

(b) If $f(t)$ has period 1, has half-wave symmetry and its Fourier series representation is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}.$$

Show that $c_n = 0$ if n is even.
Hint: $-c_n = -\int_0^1 e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t-\frac{1}{2})\, dt.$

Solution:

(a)

$$\begin{aligned}\sin(2\pi(t-\frac{1}{2})) + sin(4\pi(t-\frac{1}{2})) &= \sin(2\pi t - \pi) + \sin(4\pi t - 2\pi)\\ &= -\sin 2\pi t + \sin 4\pi t\\ &\neq -f(t)\end{aligned}$$

Therefore, $f(t)$ does not have half-wave symmetry.

$$f(t) = \frac{1}{i2} e^{i2\pi t} - \frac{1}{i2} e^{-i2\pi t} + \frac{1}{i2} e^{i4\pi t} - \frac{1}{i2} e^{-i4\pi t}$$

(b) The hint says

$$-c_n = -\int_0^1 e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t-\frac{1}{2})\, dt\,.$$

We make a change of variable $u = t - \frac{1}{2}$ in the second integral:

$$\begin{aligned}\int_0^1 e^{-2\pi i n t} f(t-\frac{1}{2})\, dt &= \int_{-1/2}^{1/2} e^{-2\pi i n(u+\frac{1}{2})} f(u)\, du\\ &= \int_{-1/2}^{1/2} e^{-2\pi i n u} e^{-2\pi i n \frac{1}{2}} f(u)\, du\\ &= e^{-\pi i n} \int_{-1/2}^{1/2} e^{-2\pi i n u} f(u)\, du\\ &= e^{-\pi i n} c_n, \quad \text{(because we can integrate over any cycle to compute } c_n).\end{aligned}$$

Thus

$$-c_n = e^{-\pi i n} c_n \,.$$

If n is even then $e^{-\pi i n} = 1$ and we have

$$-c_n = c_n \,,$$

hence

$$c_n = 0 \,.$$

A slightly different route to the same end is as follows. Again it uses the substitution $u = t - \frac{1}{2}$ in an integral.

$$\begin{aligned}
c_n &= \int_0^1 e^{-2\pi i n t} f(t)\, dt \\
&= \int_0^{1/2} e^{-2\pi i n t} f(t)\, dt + \int_{1/2}^1 e^{-2\pi i n t} f(t)\, dt \\
&= \int_0^{1/2} e^{-2\pi i n t} f(t)\, dt - \int_{1/2}^1 e^{-2\pi i n t} f(t - \frac{1}{2})\, dt \\
&= \int_0^{1/2} e^{-2\pi i n t} f(t)\, dt - \int_0^{1/2} e^{-2\pi i n (u + \frac{1}{2})} f(u)\, du \\
&= \int_0^{1/2} e^{-2\pi i n t} f(t)\, dt - e^{-\pi i n} \int_0^{1/2} e^{-2\pi i n u} f(u)\, du \,,
\end{aligned}$$

and if n is even the integrals cancel, giving $c_n = 0$.

Problem 3 (10 points) *Three independent questions on finding Fourier transforms.*

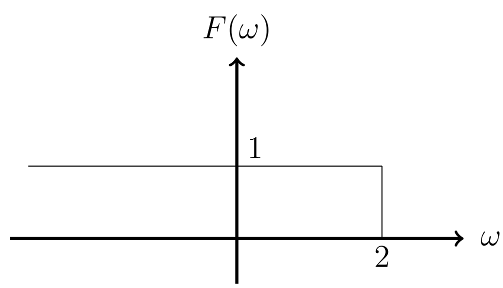
(a) (2 points) In communications theory the *analytic signal* $f_a(t)$ of a signal $f(t)$ is defined, via the Fourier transform, by

$$F_a(\omega) = \begin{cases} F(\omega), & \omega \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

where $F_a(\omega)$ is the Fourier transform of $f_a(t)$ and $F(\omega)$ is the Fourier transform of $f(t)$. For a real-valued signal $f(t)$, that is not identically zero, could the corresponding analytic signal $f_a(t)$ also be real? Why or why not?

(b) (4 points) Compute the Fourier transform of $f(x) = \cos(\pi x)\Pi(x)$, which is a half-cycle of a cosine, and $\Pi(x) = u(x + \frac{1}{2}) - u(x - \frac{1}{2})$.

(c) (4 points) The duality theorem states that that if $h(t)$ has Fourier transform $H(\omega)$, then $H(-t)$ has Fourier transform $2\pi h(\omega)$. For example, if $h(t) = \delta(t)$ then $H(\omega) = 1$ implies that $H(-t) = 1$ has Fourier transform $h(\omega) = 2\pi\delta(\omega)$. Use the duality theorem to find the inverse Fourier transform $f(t)$ of the signal in the following figure.



Solution:

(a) No, the signal cannot be real, unless it is identically zero. If $f_a(t)$ were real then its Fourier transform would have the property that $F_a(-\omega) = \overline{F_a(\omega)}$, but we are told that $F_a(\omega) = 0$ for $\omega < 0$.

(b) We can do this directly from the definition:

$$\begin{aligned} F(\omega) &= \int_{-1/2}^{1/2} \cos(\pi x) e^{-i\omega x} dx \\ &= \int_{-1/2}^{1/2} \frac{e^{\pi i x} + e^{-\pi i x}}{2} e^{-i\omega x} dx \\ &= \frac{1}{2} \int_{1/2}^{1/2} e^{-ix(\omega - \pi)} + e^{-ix(\omega+\pi)} \, dx \, . \end{aligned}$$

Integration then yields

$$F(\omega) = \frac{2\pi \cos(\omega/2)}{\pi^2 - \omega^2} \, .$$

Alternatively, using the convolution theorem

$$\begin{aligned} \mathcal{F}(\cos(\pi x)\Pi(x)) &= \frac{1}{2\pi} \mathcal{F}(\cos \pi x) \star \mathcal{F}\Pi(x) \\ &= \frac{\pi}{2\pi} (\delta(\omega - \pi) + \delta(\omega + \pi)) \star \operatorname{sinc}\left(\frac{\omega}{2}\right) \\ &= \frac{1}{2} \left(\operatorname{sinc}\left(\frac{\omega - \pi}{2}\right) + \operatorname{sinc}\left(\frac{\omega + \pi}{2}\right) \right) \\ &= \frac{\sin\left(\frac{\omega - \pi}{2}\right)}{\omega - \pi} + \frac{\sin\left(\frac{\omega+\pi}{2}\right)}{\omega + \pi} \\ &= \frac{2\pi \cos\left(\frac{\omega}{2}\right)}{\pi^2 - \omega^2}. \end{aligned}$$

(c) $F(\omega) = u(-\omega + 2)$. We know that the Fourier transform of the unit step is

$$\mathcal{F}u(t) = \pi\delta(\omega) + \frac{1}{i\omega}.$$

By the shift theorem,

$$\mathcal{F}u(t+2) = e^{i2\omega} \left(\pi\delta(\omega) + \frac{1}{i\omega} \right).$$

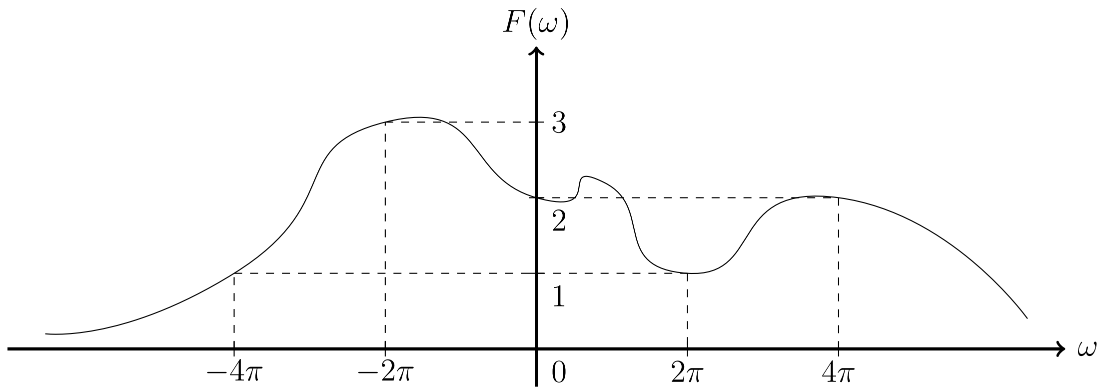
By duality,

$$f(t) = \mathcal{F}^{-1}(u(-\omega + 2)) = \frac{e^{i2t}}{2\pi} \left(\pi\delta(t) + \frac{1}{it} \right) = \frac{\delta(t)}{2} + \frac{e^{i2t}}{2\pi i t}.$$

Problem 4 (10 points) *Fourier transform values from a graph.*

The function $F(\omega)$ sketched below is the Fourier transform of an unknown function $f(x)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$



Evaluate the following integrals. Your answers should be numbers, not functions!

1. $\int_{-\infty}^{\infty} f(x)\, dx$

2. $\int_{-\infty}^{\infty} f(x) e^{-2\pi i x}\, dx$

3. $\int_{-\infty}^{\infty} f(x) e^{4\pi i x}\, dx$

4. $\int_{-\infty}^{\infty} f(x) \cos(2\pi x)\, dx$

5. $\int_{-\infty}^{\infty} f(x) \cos(2\pi x) e^{2\pi i x}\, dx$

Solution:

1.
$$\int_{-\infty}^{\infty} f(x)\,dx = F(0) = 2$$

2.
$$\int_{-\infty}^{\infty} f(x)e^{-2\pi ix}\,dx = F(2\pi) = 1$$

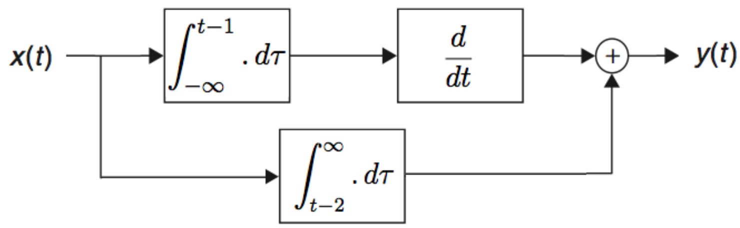
3.
$$\int_{-\infty}^{\infty} f(x)e^{4\pi ix}\,dx = F(-4\pi) = 1$$

4.
$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\cos(2\pi x)\,dx \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)e^{2\pi ix}\,dx + \int_{-\infty}^{\infty} f(x)e^{-2\pi ix}\,dx\right) \\
&= \frac{F(-2\pi) + F(2\pi)}{2} \\
&= 2
\end{aligned}$$

5.
$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\cos(2\pi x)e^{2\pi ix}\,dx \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)e^{2\pi ix}e^{2\pi ix}\,dx + \int_{-\infty}^{\infty} f(x)e^{-2\pi ix}e^{2\pi ix}\,dx\right) \\
&= \frac{F(-4\pi) + F(0)}{2} \\
&= \frac{3}{2}
\end{aligned}$$

Problem 5 (10 points)

Consider the system depicted in the following block diagram.



- (1 points) Write the input-output equations that describe this system.
- (3 points) Is this system linear? time invariant?
- (3 points) Find the impulse response.
- (3 points) Is this system stable? causal?

Solution:

- $$y(t) = \frac{d}{dt}\int_{-\infty}^{t-1} x(\tau)\,d\tau + \int_{t-2}^{\infty} x(\tau)\,d\tau = x(t-1) + \int_{t-2}^{\infty} x(\tau)\,d\tau$$
- The system is linear and time invariant because all of the subsystems are linear and time invariant.
- $$h(t) = \delta(t-1) + u(2-t)$$
- Since $\int_{-\infty}^{\infty} |h(t)|\,dt$ is not finite, the system is not stable.
  Since $h(t) \neq 0$ for some $t < 0$, the system is not causal.

Problem 6 (10 points)

The function g_h, $h > 0$, is defined as the average of the function f by

$$g_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy\,.$$

Find the Fourier transform $G_h(\omega)$ in terms of $F(\omega)$.
Hint: Make a change of variable $y = x - v$ to write $g_h(x)$ as a convolution.

Solution:

If we change variables in the integral according to the hint, $y + u = x$, then $y = x - u$, $dy = -du$ and as y goes from $x - h$ to $x + h$ u goes from h to $-h$. Thus

$$\begin{aligned} g_h(x) &= \frac{1}{2h} \int_{x-h}^{x+h} f(y)\, dy \\ &= \frac{1}{2h} \int_{h}^{-h} f(x-u)(-du) \\ &= \frac{1}{2h} \int_{-h}^{h} f(x-u)\, dy \end{aligned}$$

Now this can be written as an integral from $-\infty$ to ∞ if we cut off with Π_{2h}, i.e.

$$\begin{aligned} g_h(x) &= \frac{1}{2h} \int_{-h}^{h} f(x-u)\, dy \\ &= \frac{1}{2h} \int_{-\infty}^{\infty} \Pi_{2h}(u) f(x-u)\, du \\ &= \frac{1}{2h} (\Pi_{2h} \star f)(x) \end{aligned}$$

Now we take the Fourier transform using the convolution theorem.

$$\begin{aligned} G_h(\omega) &= \frac{1}{2h} \mathcal{F}(\Pi_{2h} \star f)(\omega) \\ &= \frac{1}{2h} \frac{\sin(h\omega)}{\omega/2} F(\omega) \end{aligned}$$