Total: 28 points

Signals and Systems

Midterm Exam

Solutions

Problem 1 (12 points)

I. (3 *points*) Evaluate the following convolutions:

- $u(t) \star e^{-t}u(t)$,
- $u(t) \star (u(t-1) - u(t-2))$.

II. (9 *points*) Consider the following three LTI systems:

- $\mathcal{S}_1$: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- $\mathcal{S}_2$: LTI system with impulse response $h_2(t) = e^{-t}u(t)$.
- $\mathcal{S}_3$: LTI system with impulse response $h_3(t) = u(-2t+2) - u(-2t+4)$.

The three systems are now interconnected as shown in Figure 1. What is the impulse response $h(t)$ of the overall LTI system (i.e. from $x(t)$ to $z(t)$)? Is the overall system stable? Causal?

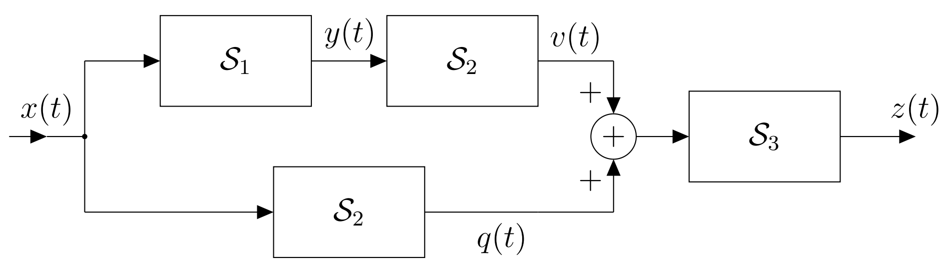


Figure 1: The LTI system in Problem 1

Solution:

I. The convolutions give the following signals:

- $u(t) \star e^{-t}u(t) = \left(\int_0^t e^{-\tau}\right) u(t) = (1 - e^{-t})u(t)$,
- $u(t) \star (u(t-1) - u(t-2)) = (t-1)u(t-1) - (t-2)u(t-2)$.

II. The first system has the following impulse response: $h_1(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$. Moreover the impulse response of the third system can be equivalently written as: $h_3(t) = u(-2t+2) - u(-2t+4) = u(-t+1) - u(-t+2) = -(u(t-1) - u(t-2))$. Thus the overall impulse response is:

$$
\begin{aligned}
&[h_1(t) \star h_2(t) + h_2(t)] \star h_3(t) \\
&= \left[u(t) \star (e^{-t}u(t)) + e^{-t}u(t)\right] \star (-(u(t-1) - u(t-2))) \\
&= -(t-1)u(t-1) + (t-2)u(t-2)
\end{aligned}
$$

The overall system is causal since $h(t) = 0$ for all $t < 0$. The overall system is not stable since $\int |h(t)|dt = \int_1^2 (t-1)dt + \int_2^\infty 1dt$ which does not converge, thus the system is unstable.

Problem 2 (8 points)

Figure 2 shows three real signals. All the signals have finite time support, i.e. the signals are zero at any time not shown in the figure.

(a) Determine which, if any, of the real signals depicted in Figure 2 have Fourier transforms that satisfy each of the following conditions (and very briefly say why):

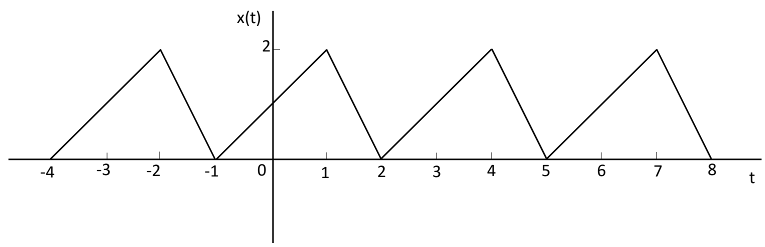
1. (1.5 *points*) $\mathcal{R}e\{X(\omega)\} \neq 0$.
2. (1.5 *points*) $\mathcal{I}m\{X(\omega)\} \neq 0$.
3. (1.5 *points*) There exists a real number such that $e^{j\alpha\omega}X(\omega)$ is real.
4. (1.5 *points*) $\int_{-\infty}^{\infty} X(\omega)d\omega = 0$.
5. (1.5 *points*) $\int_{-\infty}^{\infty} \omega X(\omega)d\omega = 0$.

(b) (1 *points*) Calculate, for the signal in Figure 2 (b) the value of $\int_{-\infty}^{\infty} \cos(2\omega)X(\omega)d\omega$. (Hint: you do not need to calculate $X(\omega)$).

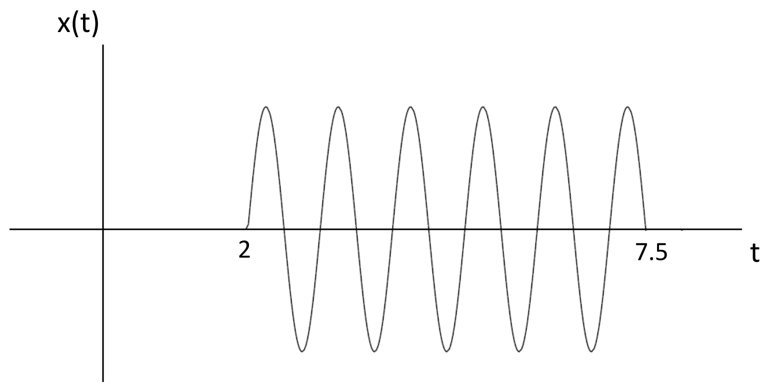
Solution:

(a) Since $x(t)$ is real, we have seen in HW.5 that $\mathcal{R}e\{X(\omega)\}$ is the Fourier transform of the even component of $x(t)$ and that $j\mathcal{I}m\{X(\omega)\}$ is the Fourier transform of the odd component of $x(t)$.

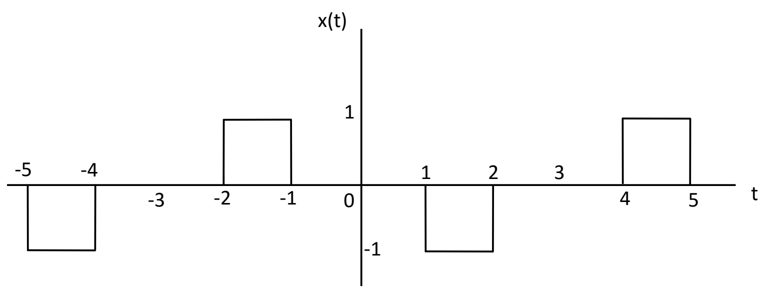
1. $\mathcal{R}e\{X(\omega)\} \neq 0$: this means that $x(t)$ has an even component, thus $x(t)$ cannot be odd. Therefore the possible signals are (a) and (b).
2. $\mathcal{I}m\{X(\omega)\} \neq 0$: this means that $x(t)$ has an odd component, thus $x(t)$ cannot be even. Therefore the possible signals are (a), (b) and (c).
3. For $e^{j\alpha\omega}X(\omega)$ to be real, its inverse Fourier transform should be even, i.e., $x(t+a)$ should be even. For the signals in (a) and (c), any shift will not make this signal even, and therefore there is no a for these signals such that $x(t+a)$ is even. For the signal in (b), a left shift by 4.75 makes the signal even, thus a possible value for a is 4.75.



(a)



(b)



(c)

Figure 2: Problem 2

4. Since

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

we can conclude that

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$$

Therefore, if $\int_{-\infty}^{\infty} X(\omega)d\omega = 0$, then $x(0) = 0$. Signals in (b) and (c) satisfy this condition.

5. Let $y(t) = \frac{dx(t)}{dt}$, then $Y(\omega) = j\omega X(\omega)$. Thus,

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega X(\omega)e^{j\omega t}d\omega$$

Moreover,

$$y(0) = \frac{j}{2\pi}\int_{-\infty}^{\infty} \omega X(\omega)d\omega$$

Therefore, if $\int_{-\infty}^{\infty} \omega X(\omega)d\omega = 0$, then $y(0) = 0$, or $\frac{dx(t)}{dt}|_{t=0} = 0$. The signals in (c) and (b) satisfy this condition.

(b) $\int_{-\infty}^{\infty} \cos(2\omega)X(\omega)d\omega = \int_{-\infty}^{\infty} \frac{e^{j2\omega}+e^{-j2\omega}}{2}X(\omega)d\omega = \pi x(2) + \pi x(-2) = 0.$

Problem 3 (8 points)

Consider a system where the output $y(t)$ is related to the input $x(t)$ by the following input-output equations:

$$y(t) = x(t)g(t) + x(t-2)g(t+3).$$

1. (3 points) If $g(t) = \delta(t)$, is this system linear? is it time invariant? is it causal? (explain why)
2. (1 point) If $g(t) = t$ is this system stable? (explain why)
3. (2 points) If $g(t) = 1$ and $x(t)$ is periodic with period $T_1 = 4$, show that $y(t)$ can be periodic with period $T_2 = 2$. Under what conditions, $y(t)$ has period $T_2 = 2$?
4. (2 points) Under the same assumptions of the previous part ($g(t) = 1$ and the conditions found for $y(t)$ to be periodic of period 2), what are the Fourier Series coefficients for $y(t)$, if we are given the Fourier series coefficients for $x(t)$?

Solution:

1. *Linear:* The system is linear because the linear combination of any two inputs gives the same linear combination of the corresponding outputs, i.e.,

$$\begin{aligned}
x_1(t) &\rightarrow y_1(t) = x_1(t)\delta(t) + x_1(t-2)\delta(t+3) \\
x_2(t) &\rightarrow y_2(t) = x_2(t)\delta(t) + x_2(t-2)\delta(t+3) \\
ax_1(t) + bx_2(t) &\rightarrow y_3(t) = [ax_1(t) + bx_2(t)]\delta(t) + [ax_1(t-2) + bx_2(t-2)]\delta(t+3) \\
&\qquad = ay_1(t) + by_2(t)
\end{aligned}$$

Time-invariant: The system is **not** time-invariant because the system definition contains time-dependent coefficients. This can be proved as follows:

$$\begin{aligned}
x_1(t) &\rightarrow y_1(t) = x_1(t)\delta(t) + x_1(t-2)\delta(t+3) \\
x_2(t) = x_1(t-t_1) &\rightarrow y_2(t) = x_2(t)\delta(t) + x_2(t-2)\delta(t+3) \\
&\qquad = x_1(t-t_1)\delta(t) + x_1(t-t_1-2)\delta(t+3)
\end{aligned}$$

But $y_1(t-t_1) = x_1(t-t_1)\delta(t-t_1) + x_1(t-t_1-2)\delta(t-t_1+3)$. Thus $y_2(t) \neq y_1(t-t_1)$ which means that $y_2(t)$ is not a shifted version of $y_1(t)$.

Causal: The system is causal because it depends on the present value of the input.

2. We have

$$y(t) = tx(t) + (t+3)x(t-2)$$

Let $x(t) = 1$, which is a bounded input, then $y(t) = 2t + 3$, since $y(t) \to \infty$ as $t \to \infty$, $y(t)$ is not a bounded signal, thus the system is not stable.

3. We have $y(t) = x(t) + x(t-2)$. For $y(t)$ to be periodic with period $T_2 = 2$, we first have to show $y(t) = y(t+2)$:

$$y(t+2) = x(t+2) + x(t+2-2) = x(t+2) + x(t)$$

But $x(t+2) = x(t-2)$ because $x(t)$ is periodic with period $T = 4$. More specifically, $x(t+2) = x(t+2-4) = x(t-2)$. Thus,

$$y(t+2) = x(t-2) + x(t) = y(t)$$

Now to ensure that this is the period, we have to have $T_2 = 2$ as the smallest real positive scalar T such that $y(t) = y(t+T)$. This is satisfied under the following condition:

- $x(t-2) + x(t) \neq$ constant signal.

4. $y(t)$ is the sum of two periodic signals of same period, then using the properties of Fourier series, the Fourier series coefficients of $y(t)$ are given by:

$$Y_k = X_k + e^{-j2\Omega_1 k} X_k$$

where $\Omega_1 = \frac{2\pi}{T_1} = \frac{\pi}{2}$. Thus,

$$\begin{aligned} Y_k &= X_k + e^{-j2\frac{\pi}{2}k} X_k = X_k + e^{-j\pi k} X_k \\ &= \begin{cases} 2X_k, & \text{if } k \text{ is even} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

To note that here Y_k corresponds to the scaled frequency $k\Omega_1$, which can be equivalently expressed as a scaled version of the frequency of $y(t)$ denoted by $\Omega_2 = \frac{2\pi}{T_2} = 2\Omega_1$. Thus, $Y_k = 2X_k$ corresponds to the frequency $k\Omega_1 = k\Omega_2 \frac{1}{2} = \frac{k}{2}\Omega_2$ where k is even.

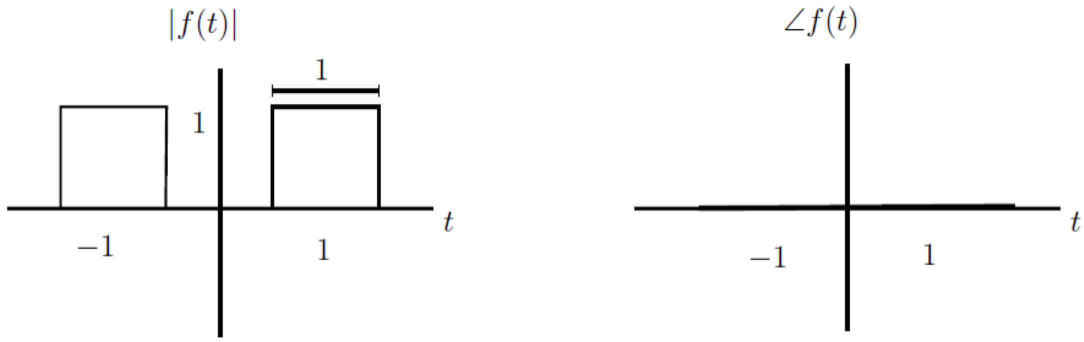
We can also derive this result as follows:

$$
\begin{aligned}
y(t) &= x(t) + x(t-2) \\
&= \sum_k X_k e^{j\Omega_1 kt} + \sum_k X_k e^{j\Omega_1 k(t-2)} \\
&= \sum_k \left(X_k + e^{-j2\Omega_1 kt} X_k\right) e^{j\Omega_1 kt} \\
&= \sum_k \left(X_k + e^{-j\pi kt} X_k\right) e^{j\Omega_1 kt} \\
&= \sum_{k,\text{ even}} 2X_k e^{j\Omega_1 kt} \\
&= \sum_{k,\text{ even}} 2X_k e^{j\frac{\Omega_2}{2}kt},\ \text{let } \ell = \frac{k}{2} \\
&= \sum_\ell 2X_{2\ell}\, e^{j\ell\Omega_2\, t}
\end{aligned}
$$

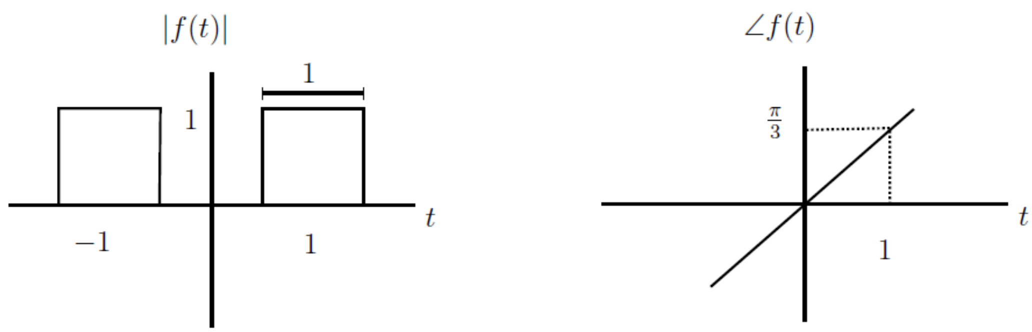
The Fourier series coefficients are given by $2X_{2\ell}$ that corresponds to frequency $\ell\Omega_2$ which is equivalent to what we obtained previously using the properties.

Problem 4 (5 points)

Write an expression for the following signals and their Fourier transforms. Recall that $f(t) = |f(t)|e^{i\angle f(t)}$, where $\angle f(t)$ is the phase of $f(t)$.



(a)

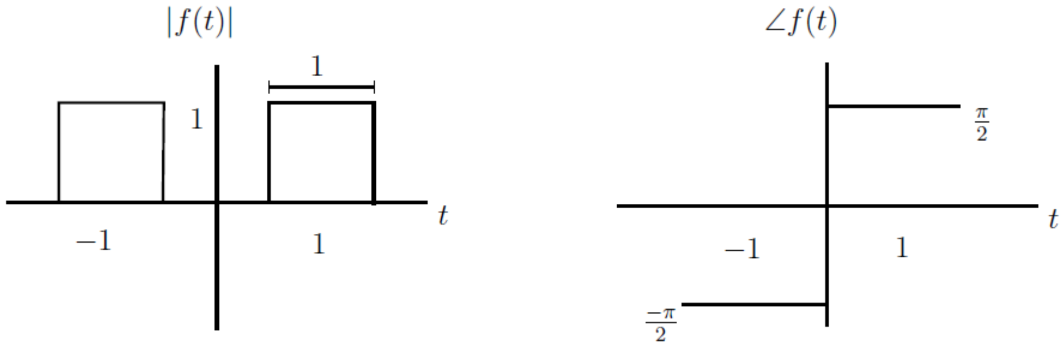


(b)

Solutions

(a) Since the signal has zero phase, we recognize this as the rectangle function convolved with two deltas, one at -1 and one at 1, i.e. a rectangle that is shifted to the left by 1 and to the right by 1. We will denote the rectangle function centred at 0 and of width 1 by $\Pi(t)$, thus we can express this signal as follows:

$$\begin{aligned} f_a(t) &= \Pi(t) * (\delta_{-1} + \delta_1) \\ &= \Pi(t+1) + \Pi(t-1). \end{aligned}$$



(c)

The Fourier transform of the rectangle function $\Pi(t)$ is the sinc function $\text{sinc}\left(\frac{1}{2}\omega\right)$. Therefore, using the Fourier transform properties, the Fourier transform of $f_a(t)$ is as follows:

$$\begin{aligned} F_a(\omega) &= e^{j\omega}\,\text{sinc}\left(\frac{1}{2}\omega\right) + e^{-j\omega}\,\text{sinc}\left(\frac{1}{2}\omega\right) \\ &= 2\,\text{sinc}\left(\frac{1}{2}\omega\right)\cos(\omega). \end{aligned}$$

(b) This is the signal in part (a), but now with a linear phase with slope $\frac{\pi}{3}$:

$$\begin{aligned} f_b(t) &= \left(\Pi(t+1) + \Pi(t-1)\right) e^{i\frac{\pi}{3}t} \\ &= f_a(t)e^{i\frac{\pi}{3}t} \end{aligned}$$

When computing the Fourier transform, we know that a linear phase in the signal domain corresponds to a shift in the Fourier domain. Since the slope of this linear phase is positive, this corresponds to shifting the Fourier transform to the right. Thus we have

$$\begin{aligned} F_b(\omega) &= F_a\left(\omega - \tfrac{\pi}{3}\right) \\ &= 2\,\text{sinc}\left(\frac{1}{2}\left(\omega - \frac{\pi}{3}\right)\right)\cos\left(\omega - \frac{\pi}{3}\right) \end{aligned}$$

(c) Here we have a purely imaginary signal. Writing it out, we have

$$\begin{aligned} f_c(t) &= e^{-i\frac{\pi}{2}}\Pi(t+1) + e^{i\frac{\pi}{2}}\Pi(t-1) \\ &= -i\Pi(t+1) + i\Pi(t-1) \end{aligned}$$

Thus, the Fourier transform of $f_c(t)$ is as follows:

$$\begin{aligned} F_c(\omega) &= -ie^{i\omega}\,\text{sinc}\left(\frac{1}{2}\omega\right) + ie^{-i\omega}\,\text{sinc}\left(\frac{1}{2}\omega\right) \\ &= 2\sin(\omega)\,\text{sinc}\left(\frac{1}{2}\omega\right) \end{aligned}$$