$x_1(t) = \text{sinc}(t) = \frac{\sin(t)}{t}$ even

$\hookrightarrow x_1(-t) = \frac{\sin(-t)}{-t} = \frac{-\sin(t)}{-t} = x_1(t)$

Problem 1 (7 points) The following three questions are not related to each other.

1. (1.5 points) Consider the following signals: $x_1(t) = \text{sinc}(t)$, $x_2(t) = r(t) - 5 + r(-t)$, and $x_3(t) = te^{-3|t|}$. Which of these signals are even? which are odd?

even: $x_1(t)$,

2. (2.5 points) Determine whether the system

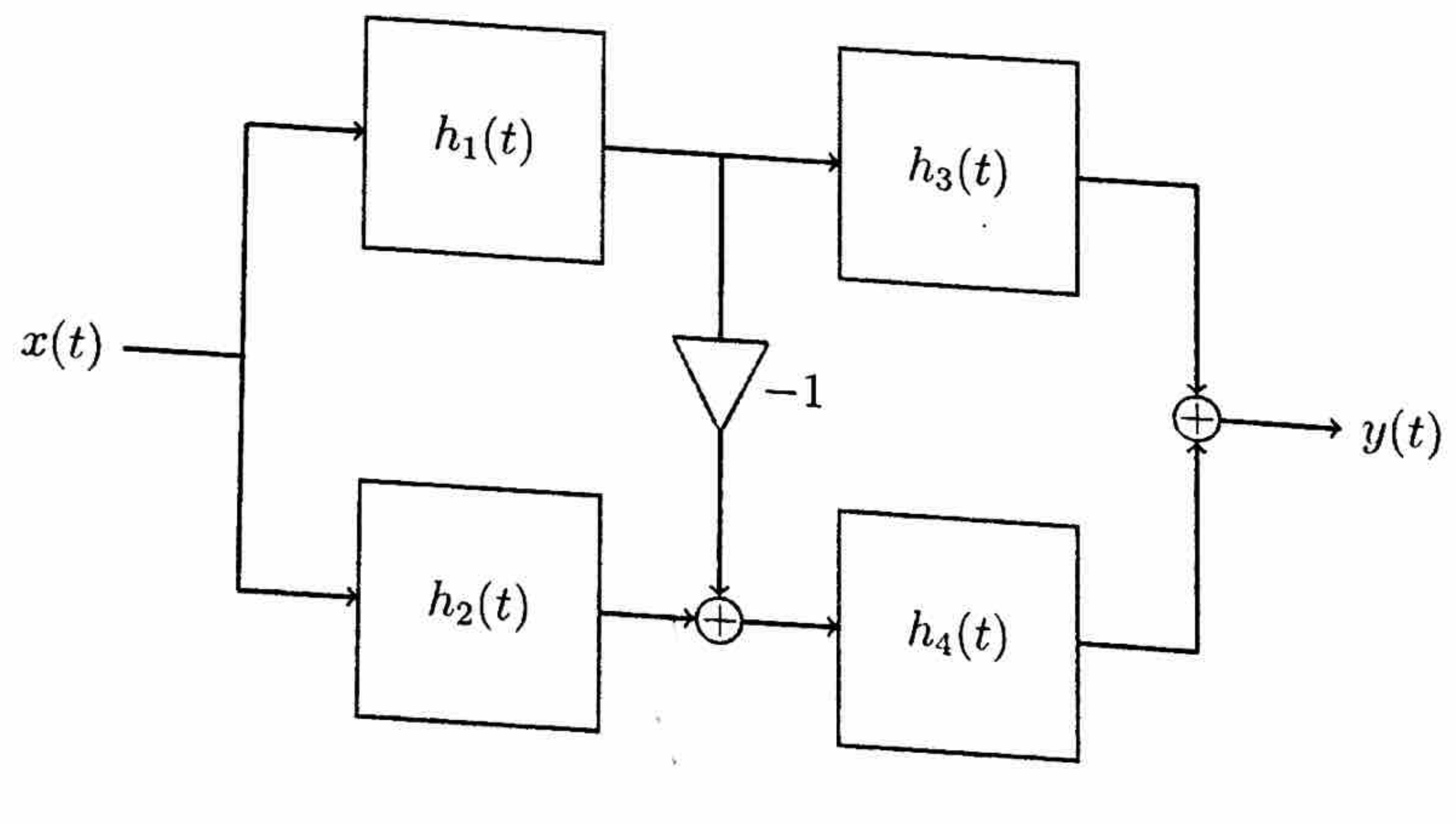
$$y(t) = \begin{cases} x(t-5) & \text{if} \quad |x(t)| \leq B \\ A|x(t)| & \text{otherwise} \end{cases},$$

where $|x(t)|$ is the magnitude of the input $x(t)$, is

(a) Causal

(b) Time invariant

3. (3 points) You are are told that the four blocks in the following block diagram represent LTI systems. Determine the expression for the impulse response of the overall system in terms of the impulse responses of the individual systems.



1. S
morning
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7.

Problem 1

1. $x_1(-t) = \frac{\sin(-t)}{-t} = \frac{-\sin(t)}{-t} = \text{sinc}(t) = x_1(t) \rightarrow$ even

$x_2(-t) = r(-t) - 5 + r(t) = r(t) - 5 + r(-t) = x_2(-t) \rightarrow$ even

$x_3(-t) = (-t)e^{-3|-t|} = -te^{-3|t|} = -(te^{-3|t|}) = -x_3(t) \rightarrow$ odd

1.5

2.

(a) $\boxed{\text{Causal}}$ since the output only depends on ~~present~~ or past values of the input.

Yes, time-invariant

2.5

(b) $x_1(t) = x(t-t_0)$

$y(t) = \begin{cases} x_1(t-5) & \text{if } |x_1(t)| \leq B \\ A|x_1(t)| & \text{otherwise} \end{cases} \rightarrow y_1(t) = \begin{cases} x(t-5-t_0) & \text{if } |x(t-t_0)| \leq B \\ A|x(t-t_0)| & \text{otherwise} \end{cases} \stackrel{?}{=} y(t-t_0) = \begin{cases} x(t-t_0-5) & |x(t-t_0)| \\ A|x(t-t_0)| & \text{otherwise} \end{cases}$

3. $h_{t_1}(t) = \underline{h_1(t) * h_3(t)}$

$h_{t_2}(t) = \overline{-(h_1(t) * h_3(t)) + h_2(t)}$

$= h_2(t) - [h_1(t) * h_3(t)]$

$$\boxed{h_t(t) = \left(h_2(t) - [h_1(t) * h_3(t)]\right) * h_4(t)}$$

1

look at solutions.

$e^{-i\frac{\pi k}{2}} = \left(e^{-i\frac{\pi}{2}}\right)^k = \left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)^k$ $\quad (-i)^k$

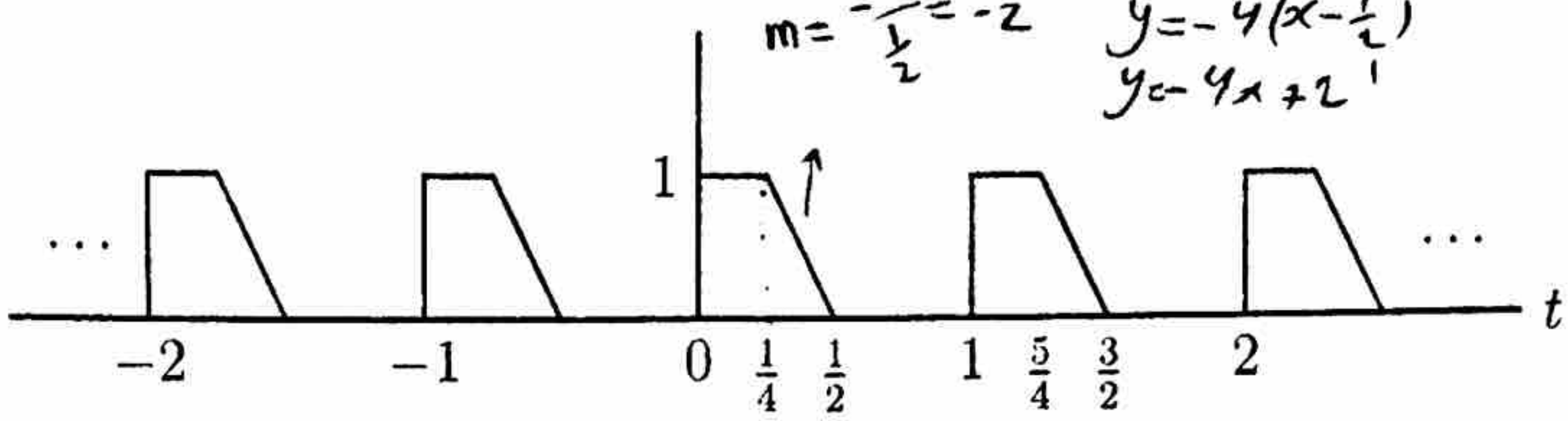
$e^{i\pi k} = e^{i\pi k} = (\cos(\pi) + i\sin(\pi))^k$ $\quad (-1)^k$

Problem 2 (7 points) (a) (4 points) Consider the signal in the following figure, that has period T=1. Calculate the Fourier Series coefficients $\{c_k\}$.

$\left(\frac{1}{2}, 0\right), \left(\frac{1}{4}, 1\right)$ $\quad m = \frac{1-0}{\frac{1}{4}-\frac{1}{2}} = \frac{1}{\frac{1-2}{4}} = \frac{1}{-\frac{1}{4}} = -4$

$m = -\frac{1}{\frac{1}{2}} = -2$ $\quad y = -4\left(x - \frac{1}{2}\right)$

$y = -4x + 2$



(b) (3 points) As we have discussed, the signal in the previous question also has as period all integer multiples of T, for instance, T=10 is also a period. What will happen if you calculate the Fourier Series coefficients, for the signal in part (a), assuming that T=10? Could you directly tell what these coefficients would be from the coefficients $\{c_k\}$ you calculated in part (a)?

4

(a) $c_k = \frac{1}{T}\int_T x(t)\, e^{-i\frac{2\pi k}{T}t}\, dt = \int_0^{\frac{1}{2}} x(t)\, e^{-i\frac{2\pi k}{T}t}\, dt = \int_0^{\frac{1}{4}} 1\cdot e^{-i\frac{2\pi k}{T}t}\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4t+2)\cdot e^{-i\frac{2\pi k}{T}t}\, dt$

$= \int_0^{\frac{1}{4}} e^{-i\frac{2\pi k}{T}t}\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} -4t e^{-i\frac{2\pi k}{T}t}\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} 2e^{-i\frac{2\pi k}{T}t}\, dt = \int_0^{\frac{1}{4}} e^{-i\frac{2\pi k}{T}t}\, dt - 4\int_{\frac{1}{4}}^{\frac{1}{2}} te^{-i\frac{2\pi k}{T}t}\, dt + 2\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-i\frac{2\pi k}{T}t}\, dt$

$= \left(\frac{1}{-i2\pi k} e^{-i2\pi k t}\Big|_0^{\frac{1}{4}} - 4\left(\frac{t}{-i2\pi k} e^{-i2\pi k t} - \frac{1}{(-i2\pi k)^2} e^{-i2\pi k t}\Big|_{\frac{1}{4}}^{\frac{1}{2}} + \frac{2}{-i2\pi k} e^{-i2\pi k t}\Big|_{\frac{1}{4}}^{\frac{1}{2}}\right.$

$= \frac{-1}{i2\pi k}\left(e^{-\frac{i\pi k}{2}} - 1\right) - 4\left[\left(\frac{1}{-i4\pi k} e^{-i\pi k} - \frac{1}{(-i2\pi k)^2} e^{-i\pi k}\right) - \left(\frac{1}{-i8\pi k} e^{-\frac{i\pi k}{2}} - \frac{1}{(-i2\pi k)^2} e^{-\frac{i\pi k}{2}}\right)\right]$

$- \frac{1}{i\pi k}\left(e^{-i\pi k} - e^{-\frac{i\pi k}{2}}\right) =$

$$c_k = \boxed{\frac{-1}{i2\pi k}\left((-i)^k - 1\right) - 4\left[\left(\frac{(-1)^k}{-i4\pi k} - \frac{(-1)^k}{(-i2\pi k)^2}\right) - \left(\frac{(-i)^k}{-i8\pi k} - \frac{(-i)^k}{(-i2\pi k)^2}\right)\right] - \frac{1}{i\pi k}\left((-1)^k - (-i)^k\right)}$$

$=$

$c_0 = \int_0^{\frac{1}{4}} 1\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} -4t\, dt + \int_{\frac{1}{4}}^{\frac{1}{2}} 2\, dt = \frac{1}{4} - 4\left(\frac{t^2}{2}\right)\Big|_{\frac{1}{4}}^{\frac{1}{2}} + 2t\Big|_{\frac{1}{4}}^{\frac{1}{2}}$

$= \frac{1}{4} - 2\left(\frac{1}{4} - \frac{1}{16}\right) + 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{4} - \frac{3}{8} + \frac{1}{2} = \frac{2-3+4}{8} = \left(\frac{3}{8}\right) \to c_0$

$\frac{3}{16}$ $\quad \frac{1}{4}$

Problem 3 (7 points)

1. (2 points) Calculate the Fourier transform of the signal $\Pi(at-b)$, where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$ as a function of the parameters a and b.

2. (2 points) Let $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ be the derivative of the signal x(t), where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$. Calculate the Fourier transform of x(t).

3. (3 points) Calculate the following integral

$0.5\Pi(\frac{1}{2}(t-5))$

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau$$

(1.) $x_1(t) = \Pi(t) \longleftrightarrow X(\omega) = \mathrm{sinc}(\pi f)$

$x_2(t) = \Pi(at) \longleftrightarrow \frac{1}{|a|} \mathrm{sinc}(\frac{\pi f}{a})$

$x_3(t) = \Pi(a(t - \frac{b}{a})) \longleftrightarrow \boxed{\frac{1}{|a|} \mathrm{sinc}(\frac{\pi f}{a}) \cdot e^{-j\omega \frac{b}{a}}}$

(2)

(2) $y(t) = \frac{d}{dt} x(t) \longleftrightarrow (j\omega) X(\omega) = Y(\omega)$?

$Y(\omega) = \mathrm{sinc}(\pi f) . e^{-j\omega 2} - \frac{1}{2} \cdot \frac{1}{|\frac{1}{2}|} \mathrm{sinc}(\frac{\pi f}{1/2}) . e^{-j\omega . 5}$

$= e^{-2j\omega} \mathrm{sinc}(\pi f) - e^{-5j\omega} \mathrm{sinc}(2\pi f) = (j\omega) X(\omega)$

$\Rightarrow \boxed{X(\omega) = \frac{1}{j\omega} [e^{-2j\omega} \mathrm{sinc}(\pi f) - e^{-5j\omega} \sin(2\pi f)]}$

(2)

(3.) $y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau}{18\tau} \cdot \frac{\sin 4\tau}{4\tau} . (18)(4) . e^{i11\tau} d\tau$ $\to t = 11$

$y(t) = (18)(4) \int_{-\infty}^{\infty} \mathrm{sinc}(18\tau) . \mathrm{sinc}(4\tau) e^{i11\tau} d\tau$ $\to X(\omega) = \underbrace{\mathrm{sinc}(18\omega)}_{X_1(\omega)} . \underbrace{\mathrm{sinc}(4\omega)}_{X_2(\omega)}$

$X(t) = x_1(t) * x_2(t)$

$x_1(t) = \frac{1}{36} \Pi(\frac{t}{36})$

$x(t) = \frac{1}{36} \Pi(\frac{t}{36}) * \frac{1}{8} \Pi(\frac{t}{8}) =$

$x_2(t) = \frac{1}{8} \Pi(\frac{t}{8})$

(3)

$y(t) = (18)(4) 2\pi . \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{sinc}(18\tau) \mathrm{sinc}(4\tau) e^{i11\tau} d\tau = (18)(4)(2\pi) . x(11).$

$$y(t) = (18)(4)2\pi \cdot x(11) =$$

$$= \cancel{(18)}\cancel{(4)}(8\pi) \cdot \frac{1}{\cancel{36}} \pi\left(\frac{11}{36}\right) * \frac{1}{\cancel{8}} \pi\left(\frac{11}{8}\right)$$

$\cancel{2}$ $\quad$ 2

$$= \pi\left[\Pi\left(\frac{11}{36}\right) * \frac{1}{2} \Pi\left(\frac{11}{8}\right) \right]$$

6/7

Problem 4 (7 points) Sometimes we work with systems that take as input two signals, say $f(t)$ and $g(t)$ and produce at their output one signal, say $y(t)$. One way of analyzing such systems is by assuming they take as input a 2×1 vector $x(t)$ that has as elements the signals $f(t)$ and $g(t)$; we then apply the definitions for linearity and time invariance on the vector input $x(t)$.

Consider a system that takes as input two real signals $f(t)$ and $g(t)$ and calculates as output their inner product $y(t)$ defined as

$$y(t) = (f, g) = \int_{-\infty}^{\infty} f(t)g(t)\, dt$$

Recall that the time reverse of a signal $x(t)$ is the signal $x(-t)$, and the time shifted version of $x(t)$ by some constant t_0 is the signal $x(t - t_0)$.

(a) If both $f(t)$ and $g(t)$ are time reversed, what happens to their inner product?

(b) Assume that only one of $f(t)$ and $g(t)$ is time reversed, does the outcome depend on which one was reversed or no?

$\int x_1(\tau)x_2(t-\tau)d\tau$

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product? $x(t)$ $x(-t)$ $x_1 * x_2$ $x_1(t)$ $x_1(t-t_0)$ $x(t)$ $\int x(t)\delta(t-t_1)\,dt$ $x(t_1)$

(d) Assume that you can use as blocks the following systems: a block that takes as input a signal and time reverses it, a block that takes as input 2 signals $x_1(t)$ and $x_2(t)$ and outputs the signal $y(t) = x_1(t) * x_2(t)$ that is their convolution, a block that takes as input a signal and delays it by a fixed amount t_0 we can select, and a block that takes as input a signal $x(t)$ and outputs the constant value $\int_{\infty}^{\infty} x(t)\delta(t - t_1)dt$ for a constant t_1 we can select. Can you connect (some of) these blocks to create a system that takes as input two signals and outputs their inner product value? x_1, x_2 $\int x_1(t)x_2(t)\,dt$

(e) Is the system that implements the inner product time invariant? Is it linear?

(a) $y(t) = (f,g) = \int_{-\infty}^{\infty} f(-t)g(-t)\,dt = \int_{\infty}^{-\infty} f(\tau)g(\tau)\cdot -d\tau = -\int_{\infty}^{-\infty} f(\tau)g(\tau)\,d\tau = \int_{-\infty}^{\infty} f(\tau)g(\tau)\,d\tau$

$\tau = -t$ $\quad t \to \infty$, $\tau = -\infty$ $\quad t \to -\infty$, $\tau \to \infty$

$d\tau = -dt$

$-d\tau = dt$

(stays the same)

(b) $y(t) = \int_{-\infty}^{\infty} f(-t)g(t)\,dt$ vs. $\int_{-\infty}^{\infty} f(t)g(-t)\,dt$ $\quad f(-t)g(t) = f(t)g(-t)$ or $f(-t)g(t) = f(t)$, $-f(t)g(t)$, $g(-t)$, $-f(t)g(t)$

if both the signals $f(t)$ & $g(t)$ are even, then we get the same outcome.

or both are odd.

++

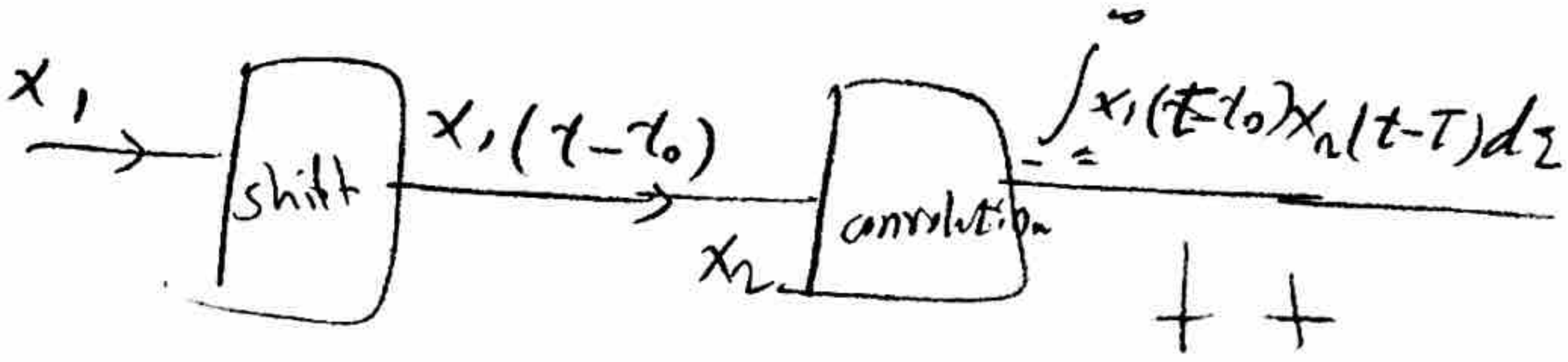
(c) $y(t) = \int_{-\infty}^{\infty} f(t-t_0)\, g(t-t_0)\, dt = \int_{-\infty}^{\infty} f(\tau)\, g(\tau)\, d\tau \rightarrow$ stays the same.

$\tau = t - t_0 \rightarrow d\tau = dt$

$t \to \infty,\ \tau = \infty$

$t \to -\infty,\ \tau = -\infty$

(d)



$$x(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$$

(e) $y(t) = \int_{-\infty}^{\infty} f(t)\, g(t)\, dt$

(I) linearity: $y_1(t) =$

$w = \alpha x_1(t) + \beta x_2(t) \rightarrow \alpha \begin{bmatrix} f_1(t) \\ g_1(t) \end{bmatrix} + \beta \begin{bmatrix} f_2(t) \\ g_2(t) \end{bmatrix} = \begin{bmatrix} \alpha f_1(t) + \beta f_2(t) \\ \alpha g_1(t) + \beta g_2(t) \end{bmatrix}$

$y(t) = \int_{-\infty}^{\infty} [\alpha f_1(t) + \beta f_2(t)][\alpha g_1(t) + \beta g_2(t)]\, dt = \int_{-\infty}^{\infty} (\alpha^2 f_1 g_1 + \alpha\beta f_1 g_2 + \alpha\beta f_2 g_1 + \beta^2 f_2 g_2)\, dt$

$\alpha y_1(t) + \beta y_2(t) = \alpha \int_{-\infty}^{\infty} f_1(t) g_1(t) + \beta \int_{-\infty}^{\infty} f_2(t) g_2(t)$ Not linear

time invariant: $x_1(t) = x(t-t_0) = \begin{bmatrix} f(t-t_0) \\ g(t-t_0) \end{bmatrix}$

$y_1(t) = \int_{-\infty}^{\infty} f(t-t_0)\, g(t-t_0)\, dt = \int_{-\infty}^{\infty} f(\tau)\, g(\tau)\, d\tau \stackrel{!}{=} y(t-t_0) = \int_{-\infty}^{\infty} f(t-t_0)\, g(t-t_0) = \int_{-\infty}^{\infty} f(\tau)\, g(\tau)\, d\tau$

Yes, time invariant

1 Properties of Fourier Series

$x(t)$ and $y(t)$ are periodic signals of period T. ($\omega_0 = \frac{2\pi}{T}$)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}, \quad d_k = \frac{1}{T}\int_T y(t)e^{-jk\omega_0 t}\,dt$$

Property	Signal	k^{th} Fourier coefficient
	$x(t)$ $y(t)$	c_k d_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha c_k + \beta d_k$
Time-Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} c_k$
Conjugation	$x^*(t)$	c^*_{-k}
Time-Reversal	$x(-t)$	c_{-k}
Time-Scaling	$x(\alpha t), \alpha > 0$ Period : $\frac{T}{\alpha}$	c_k
Conjugate-Symmetry	$x(t)$ is real	$c_k = c^*_{-k}$
Even-Odd Signals	$x(t)$ is real and even $x(t)$ is real and odd	c_k is real and even c_k is purely imaginary and odd

Parsevals Relation for Periodic Signals: $\frac{1}{T}\int\limits_T |x(t)|^2 dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

2 Fourier Transform Formulas

Fourier transform formulas (using ω):

Synthesis equation (Inverse Fourier Transform): $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Analysis equation (Fourier Transform): $X(\omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Fourier transform formulas (using f):

Synthesis equation (Inverse Fourier Transform): $x(t) = \int\limits_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$

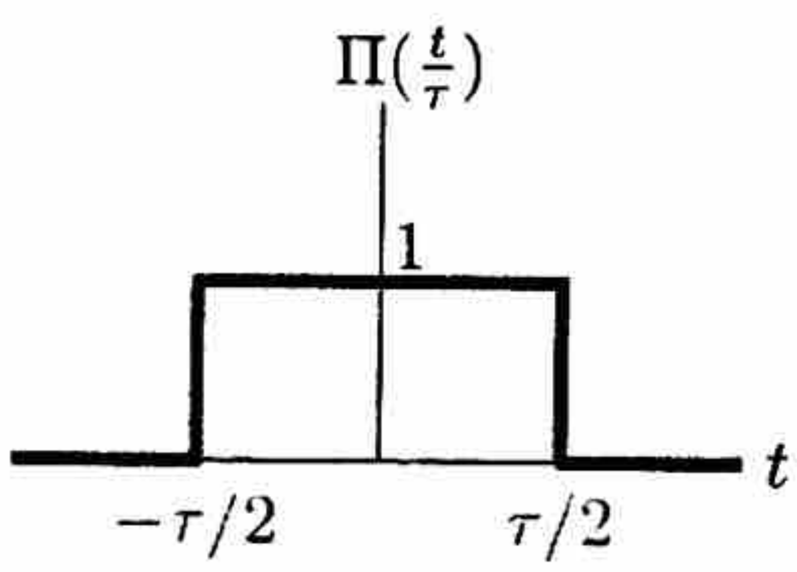
Analysis equation (Fourier Transform): $X(f) = \int\limits_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

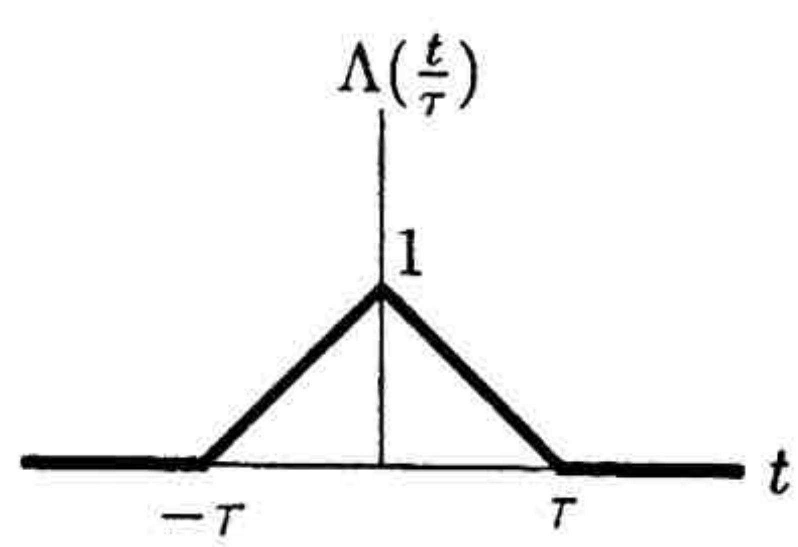
3 Fourier Transform Properties

Property	Signal	Fourier Transform
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$
Conjugate symmetry	$x(t)$ is real	$X^*(\omega) = X(-\omega)$
Conjugate anti-symmetry	$x(t)$ is purely imaginary	$X^*(\omega) = -X(-\omega)$
Even and real signal	$x(-t) = x(t)$	$\mathrm{Im}\{X(\omega)\} = 0$
Odd and real signal	$x(-t) = -x(t)$	$\mathrm{Re}\{X(\omega)\} = 0$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Modulation Property	$x(t)\cos(\omega_0 t)$	$\frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$
Differentiation in time	$\frac{d^n}{dt^n}[x(t)]$	$(j\omega)^n X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$

Parseval's theorem: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

4 Fourier Transform pairs





We define, $\mathbf{sinc}(x) := \frac{\sin(x)}{x}$

Name	Signal	Fourier Transform
Rectangular pulse	$x(t) = A\,\Pi(t/\tau)$	$X(\omega) = A\tau\,\mathrm{sinc}(\frac{\omega\tau}{2})$
Triangular pulse	$x(t) = A\,\Lambda(t/\tau)$	$X(\omega) = A\tau\,\mathrm{sinc}^2(\frac{\omega\tau}{2})$
Right-sided exponential	$x(t) = e^{-at}u(t)$	$X(\omega) = \frac{1}{a+j\omega}$
Two-sided exponential	$x(t) = e^{-a\lvert t\rvert}$	$X(\omega) = \frac{2a}{a^2+\omega^2}$
Unit impulse	$x(t) = \delta(t)$	$X(\omega) = 1$
Sinc function	$x(t) = \mathrm{sinc}(t)$	$X(\omega) = \Pi(\frac{\omega}{2\pi})$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(\omega) = 2\pi\delta(\omega)$
Unit-step function	$x(t) = u(t)$	$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$