Signals and Systems

Midterm Exam

8:00 am - 10:00 am, November 1, 2016

NAME: [redacted]

This exam has 4 problems, for a total of 28 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
The Fourier series and transform tables are provided in the last two pages.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	7	7
2	5	7
3	2.5	7
4	5	7
Total	19.5	28

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Problem 1 (7 points) The following three questions are not related to each other.

1. (1.5 points) Consider the following signals: $x_1(t) = \mathrm{sinc}(t)$, $x_2(t) = r(t) - 5 + r(-t)$, and $x_3(t) = te^{-3|t|}$. Which of these signals are even? which are odd?

2. (2.5 points) Determine whether the system

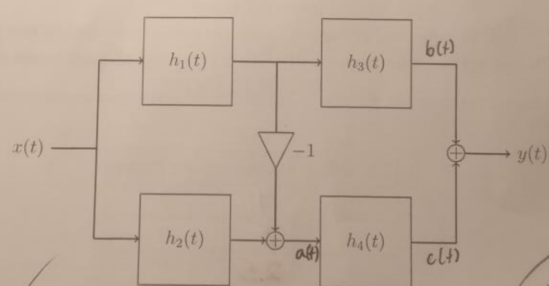
$$y(t) = \begin{cases} x(t-5) & \text{if} \quad |x(t)| \le B \\ A|x(t)| & \text{otherwise} \end{cases},$$

   where $|x(t)|$ is the magnitude of the input $x(t)$, is

   (a) Causal

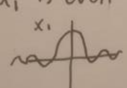
   (b) Time invariant

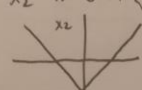
3. (3 points) You are are told that the four blocks in the following block diagram represent LTI systems. Determine the expression for the impulse response of the overall system in terms of the impulse responses of the individual systems.

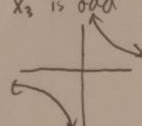


1.5

1. x_1 is even x_2 is even x_3 is odd







2. a. It is causal because the output doesn't look at future values of x

   b. It is time invariant because a shifted output is equal to the output given a shifted in

$$y(t-t_0) = \begin{cases} x(t-t_0-5) & \text{if } |x(t-t_0)| \le B \\ A|x(t-t_0)| & \text{otherwise} \end{cases} \qquad x(t-t_0) \Rightarrow y(t) = \begin{cases} x(t-t_0-5) & \text{if } |x(t-t_0)| \le B \\ A|x(t-t_0)| & \text{otherwise} \end{cases}$$

=

2.5

3. $a(t) = h_2 - h_1$

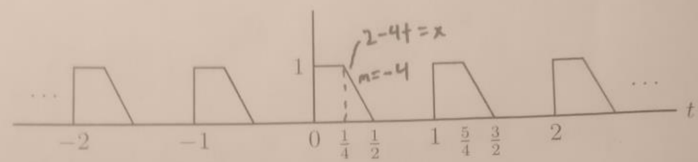
$b(t) = h_1 * h_3$

$c(t) = a(t) * h_4 = (h_2 - h_1) * h_4$

$y(t) = b(t) + c(t) = (h_1 * h_3) + ((h_2 - h_1) * h_4)$

3

Problem 2 (7 points) (a) (4 points) Consider the signal in the following figure, that has period T=1. Calculate the Fourier Series coefficients $\{c_k\}$.



(b) (3 points) As we have discussed, the signal in the previous question also has as period all integer multiples of T, for instance, T=10 is also a period. What will happen if you calculate the Fourier Series coefficients, for the signal in part (a), assuming that T=10? Could you directly tell what these coefficients would be from the coefficients $\{c_k\}$ you calculated in part (a)?

a. $c_k = \frac{1}{T}\int_0^1 x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\left(\int_0^{1/4} e^{-jk\omega_0 t}dt + \int_{1/4}^{1/2}(2-4t)e^{-jk\omega_0 t}dt\right)$

$\omega_0 = \frac{2\pi}{T} = 2\pi$

$\int_0^{1/4} e^{-jk2\pi t}dt = \frac{e^{-j2\pi k t}}{-j2\pi k}\Big]_0^{1/4} = \frac{e^{-j\frac{\pi}{2}k}-1}{-j2\pi k}$

$\int_{1/4}^{1/2}(2-4t)e^{-jk2\pi t}dt = 2\int_{1/4}^{1/2} e^{-jk2\pi t}dt - 4\int_{1/4}^{1/2} te^{-jk2\pi t}dt$

$\int_{1/4}^{1/2} e^{-jk2\pi t}dt = \frac{e^{-j\pi k} - e^{-j\frac{\pi}{2}k}}{-j2\pi k}$

$\int_{1/4}^{1/2} te^{-jk2\pi t}dt = \frac{te^{-jk2\pi t}}{-jk2\pi}\Big]_{1/4}^{1/2} - \int_{1/4}^{1/2}\frac{e^{-jk2\pi t}}{-jk2\pi}dt = \frac{e^{-jk\pi}}{-jk4\pi} + \frac{e^{-jk\frac{\pi}{2}}}{+jk8\pi} + \frac{e^{-jk\pi}}{+k^2 4\pi^2} + \frac{e^{-jk}}{-k^2 4}$

$\int_{1/4}^{1/2}(2-4t)e^{-jk2\pi t}dt = 2\left(\frac{e^{-j\pi k} - e^{-j\frac{\pi}{2}k}}{-j2\pi k}\right) - 4\left(\quad\right)$

$= -\frac{\cancel{e^{-j\pi k}} - e^{-j\frac{\pi}{2}k}}{+j\pi k} + \frac{\cancel{e^{-j\pi k}}}{jk\pi} - \frac{e^{-jk\frac{\pi}{2}}}{jk2\pi} - \frac{e^{-jk\pi}}{k^2\pi^2} + \frac{e^{-jk\frac{\pi}{2}}}{k^2\pi^2}$

$= \frac{2e^{-j\frac{\pi}{2}k}}{j2\pi k} - \frac{e^{-j\frac{\pi}{2}k}}{j2\pi k} - \frac{e^{-jk\pi}}{k^2\pi^2} + \frac{e^{-jk\frac{\pi}{2}}}{k^2\pi^2} = \frac{e^{-j\frac{\pi}{2}k}}{j2\pi k} - \frac{e^{-jk\pi}}{\pi^2 k^2} + \frac{e^{-j\frac{\pi}{2}k}}{\pi^2 k^2}$

CONT. ON NEXT PAGE

$$C_k = \frac{e^{-j\frac{\pi}{2}k}}{j2\pi k} + \frac{e^{-j\frac{\pi}{2}k}}{\pi^2 k^2} - \frac{e^{-j\pi k}}{\pi^2 k^2}$$

$$= \frac{j\pi k e^{-j\frac{\pi}{2}k}}{-2\pi^2 k^2} + \frac{-2e^{-j\frac{\pi}{2}k}}{-2\pi^2 k^2} - \frac{e^{-j\pi k}}{\pi^2 k^2}$$

$$= \left(\frac{-2+j\pi k}{-2\pi^2 k^2}\right) e^{-j\frac{\pi}{2}k} - \frac{1}{\pi^2 k^2} e^{-j\pi k}$$

4

$$= \left(\frac{2-j\pi k}{2\pi^2 k^2}\right) e^{-j\frac{\pi}{2}k} - \left(\frac{1}{\pi^2 k^2}\right) e^{-j\pi k}$$

C_k is not defined for $k=0$ so let us do that separately

$$c_0 = \frac{1}{T}\int_0^1 x(t)\,dt = \int_0^{1/2} x(t) = \frac{1}{4} + \frac{1}{4}\cdot 1 \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

good

$$C_k = \begin{cases} \frac{3}{8} & \text{if } k=0 \\ \left(\frac{2-j\pi k}{2\pi^2 k^2}\right) e^{-j\frac{\pi}{2}k} - \left(\frac{1}{\pi^2 k^2}\right) e^{-j\pi k} & \text{if } k \neq 0 \end{cases}$$

$$\omega_0' = \frac{2\pi}{10}$$

b. In the formula for $c_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t}\,dt$

If we used $T=10$, $\frac{1}{T}$ would be 10 times smaller but $\int_0^T x(t) e^{-jk\omega_0 T}\,dt$ would be 10 times bigger. why??

These would essentially cancel each other out and we would end up with the same Fourier coefficients.

Thus the coefficients would be the same C_k as from part (a)

1

look at solutions.

Problem 3 (7 points)

1. (2 points) Calculate the Fourier transform of the signal $\Pi(at-b)$, where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$ as a function of the parameters a and b.

2. (2 points) Let $y(t) = \Pi(t-2) - 0.5\Pi(\frac{t-5}{2})$ be the derivative of the signal x(t), where $\Pi(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{else} \end{cases}$. Calculate the Fourier transform of x(t).

3. (3 points) Calculate the following integral

$$y(t) = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{i11\tau} d\tau$$

1.

$\Pi(t/\lambda) \overset{F}{\leftrightarrow} \lambda \,\text{sinc}\left(\frac{\omega\lambda}{2}\right)$

$\lambda = \frac{1}{a}$

$\Pi(at) \overset{F}{\leftrightarrow} \frac{1}{a} \,\text{sinc}\left(\frac{\omega}{2a}\right)$

$\Pi(at-b) \overset{F}{\leftrightarrow} \boxed{\frac{1}{a} \,\text{sinc}\left(\frac{\omega}{2a}\right) e^{\cancel{-j\omega b}}}$ $e^{-j\omega\frac{b}{a}}$ 1

$\Pi(a(t-\frac{b}{a}))$

2. $\frac{dx(t)}{dt} = y(t)$

$\Pi(t-2) \overset{F}{\leftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right)e^{-j\omega 2}$

$\Pi\left(\frac{t}{2}\right) \overset{F}{\leftrightarrow} 2\,\text{sinc}\left(\frac{2\omega}{2}\right) = 2\,\text{sinc}(\omega)$

$\Pi\left(\frac{t}{2}-\frac{5}{2}\right) \overset{F}{\leftrightarrow} 2\,\text{sinc}(\omega)e^{-j\frac{5}{2}\omega}$

$0.5\Pi\left(\frac{t}{2}-\frac{5}{2}\right) \overset{F}{\leftrightarrow} \text{sinc}(\omega)e^{-j\frac{5}{2}\omega}$

$\Pi(t-2) - 0.5\,\Pi\left(\frac{t-5}{2}\right) \overset{F}{\leftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right)e^{-j2\omega} - \text{sinc}(\omega)\, e^{-j\frac{5}{2}\omega}$ 1.5

$\frac{d}{dt}(x(t)) \overset{F}{\leftrightarrow} \text{sinc}\left(\frac{\omega}{2}\right)e^{-j2\omega} - \text{sinc}(\omega)e^{-j\frac{5}{2}\omega}$ $e^{-j5\omega}$

$x(t) \overset{F}{\leftrightarrow} \frac{1}{j\omega}\left(\quad \right) = \boxed{-\frac{j}{\omega}\left(\text{sinc}\left(\frac{\omega}{2}\right)e^{-j2\omega} - \text{sinc}(\omega)e^{-j\frac{5}{2}\omega}\right)}$

3. $y = \int_{-\infty}^{\infty} \frac{\sin 18\tau \sin 4\tau}{\tau^2} e^{j11\tau} d\tau$ replacing τ with t to make easier to write

$$= \int_{-\infty}^{\infty} \frac{1}{t^2} \left(\frac{e^{j18t} - e^{-j18t}}{2j} \right) \left(\frac{e^{j4t} - e^{-j4t}}{2j} \right) e^{j11t} dt$$

$$= \int_{-\infty}^{\infty} -\frac{1}{4t^2} e^{j11t} \left(e^{j22t} - e^{-j14t} - e^{j14t} + e^{-j22t} \right) dt$$

0

$$= \int_{-\infty}^{\infty} -\frac{1}{4t^2} \left(e^{j33t} - e^{-j3t} - e^{j25t} + e^{-j11t} \right) dt$$

$$= -\frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{t^2} \left(e^{j33t} - e^{-j3t} - e^{-j25t} + e^{-j11t} \right) dt$$

$$\int \frac{1}{t^2} e^{j\omega t} dt = -\frac{1}{t} e^{j\omega t} + \int \frac{1}{t} j\omega e^{j\omega t}$$

$u = e^{j\omega t}$ $\quad v = -\frac{1}{t}$

$du = j\omega e^{j\omega t}$ $\quad dv = \frac{1}{t^2} dt$

I'm not sure how to do $\int \frac{1}{t^2} e^{t}$ but if I did, I would further simplify

$$-\frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{t^2} \left(e^{j33t} - e^{-j3t} - e^{-j25t} + e^{-j11t} \right)$$

$$y = \int A \operatorname{sinc}(18\tau) \operatorname{sinc}(4\tau) e^{j\ldots}$$ $(4 \cdot 18)$

$$= \int_{-\infty}^{\infty} -\frac{1}{4t^2} e^{j11t} \left(2\cos(22t) - 2\cos(14t) \right)$$

Problem 4 (7 points) Sometimes we work with systems that take as input two signals, say $f(t)$ and $g(t)$ and produce at their output one signal, say $y(t)$. One way of analyzing such systems is by assuming they take as input a 2×1 vector $x(t)$ that has as elements the signals $f(t)$ and $g(t)$; we then apply the definitions for linearity and time invariance on the vector input $x(t)$.

Consider a system that takes as input two real signals $f(t)$ and $g(t)$ and calculates as output their inner product $y(t)$ defined as

$$y(t) = \langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)\,dt$$

$x_1(t) * x_2(-t) = \int x_1(\tau)\, x_2(-t+\tau)$

$x_2(-t) * x_1(t) = \int x_2(-\tau)\, x_1(t-\tau)$

~~[illegible]~~ $x_2(-t+t_0) * x_1(t) = \int x_2(-\tau + t_0) \ldots$

Recall that the time reverse of a signal $x(t)$ is the signal $x(-t)$, and the time shifted version of $x(t)$ by some constant t_0 is the signal $x(t - t_0)$.

(a) If both $f(t)$ and $g(t)$ are time reversed, what happens to their inner product?

(b) Assume that only one of $f(t)$ and $g(t)$ is time reversed, does the outcome depend on which one was reversed or no?

(c) If both $f(t)$ and $g(t)$ are shifted by the same amount, what happens to their inner product?

(d) Assume that you can use as blocks the following systems: a block that takes as input a signal and time reverses it, a block that takes as input 2 signals $x_1(t)$ and $x_2(t)$ and outputs the signal $y(t) = x_1(t) * x_2(t)$ that is their convolution, a block that takes as input a signal and delays it by a fixed amount t_0 we can select, and a block that takes as input a signal $x(t)$ and outputs the constant value $\int_{-\infty}^{\infty} x(t)\delta(t - t_1)dt$ for a constant t_1 we can select. Can you connect (some of) these blocks to create a system that takes as input two signals and outputs their inner product value?

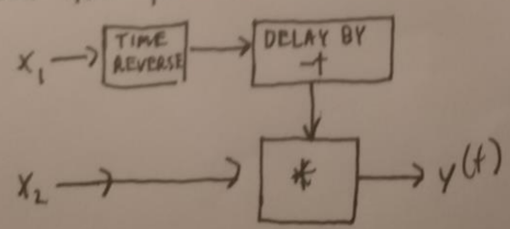
(e) Is the system that implements the inner product time invariant? Is it linear?

a. Their inner product stays the same because negative time values create the old positive time products & positive time values create the old negative time products. or $f(-t)$ will be multiplied by $g(-t)$?

b. No it doesn't matter because no matter what, $f(t)$ will be multiplied by $g(-t)$

c. Their inner product ~~[illegible]~~ doesn't change because $f(t)$ will still be multiplied

d.



$x_1(-t+t_0) * x_2(t) = \int_{-\infty}^{\infty} x_1(t_0-\tau)\, x_2$

if $t_0 = t$

$x_1(-t+t_0) * x_2(t) = y(t)$

e.

This system is time-invariant because the output is a constant signal. If a shifted input is given, the output will not change so that can be thought of as a shifted constant signal.

$$x_1(t) \quad x_2(t)$$

$$x_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \quad x_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$$

$$\alpha x_1 + \beta x_2 = \begin{pmatrix} \alpha f_1 \\ \alpha g_1 \end{pmatrix} + \begin{pmatrix} \beta f_2 \\ \beta g_2 \end{pmatrix} = \begin{pmatrix} \alpha f_1 + \beta f_2 \\ \alpha g_1 + \beta g_2 \end{pmatrix}$$

$$y(\alpha x_1 + \beta x_2) = \int_{-\infty}^{\infty} (\alpha f_1 + \beta f_2)(\alpha g_1 + \beta g_2)\,dt = \int_{-\infty}^{\infty} \alpha^2 f_1 g_1 + \alpha\beta(f_1 g_2 + f_2 g_1) + \beta^2 f$$

$$y(\alpha x_1) = \int_{-\infty}^{\infty} (\alpha f_1)(\alpha g_1)\,dt = \int_{-\infty}^{\infty} \alpha^2 f_1 g_1\,dt$$

$$y(\beta x_2) = \int_{-\infty}^{\infty} (\beta f_2)(\beta g_2)\,dt = \int_{-\infty}^{\infty} \beta^2 f_2 g_2\,dt$$

$$y(\alpha x_1) + y(\beta x_2) = \int_{-\infty}^{\infty} \alpha^2 f_1 g_1 + \beta^2 f_2 g_2\,dt$$

$\neq$

Because $y(\alpha x_1 + \beta x_2) \neq y(\alpha x_1) + y(\beta x_2)$, this system is not linear