Total: 25 points

EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, November 15, 2017

NAME: ____________________ **UID**: ____________________

This exam has 3 problems, for a total of 25 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1		10
2		8
3		7
Total		25

Extra Pages: _________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (10 points) The following questions are not related.

1. (3 points) Consider a system where the output $w(t)$ depends on the input $v(t)$ through the equation: $w(t) = \cos(v(t))$. Is this system time invariant? is it linear? is it stable? Explain why or why not.

Solution:

Time Invariance: Let $y(t) = v(t - t_d)$. Then the output of the system when $y(t)$ is the input is $\cos(y(t)) = \cos(v(t - t_d)) = w(t - t_d)$. Thus the system is $\boxed{\text{time-invariant}}$.

Linearity: Let $v_1(t) = \pi/2$ $v_2(t) = -\pi/2$. Then we have that

$$\begin{aligned} v_1(t) = \pi/2 &\rightarrow w_1(t) = \cos(\pi/2) = 0 \\ v_2(t) = -\pi/2 &\rightarrow w_2(t) = \cos(-\pi/2) = 0 \\ v_1(t) + v_2(t) = 0 &\rightarrow \cos(0) = 1. \end{aligned}$$

Thus, the system is $\boxed{\text{non-linear}}$ since $v_1(t) + v_2(t) \not\rightarrow w_1(t) + w_2(t)$.

Stability: Note that For any $v(t)$ such that $|v(t)| < \infty$, we know that $|w(t)| = |\cos(v(t))| \leq 1 < \infty$. Therefore, the system is $\boxed{\text{BIBO stable}}$.

2. (3 points) Find the time domain representation for a signal with Fourier Transform:

$$X(\omega) = \cos(a\omega)\sin(b\omega)\text{sinc}\left(\frac{\omega}{2\pi}\right).$$

Hint: You can easily find the inverse Fourier Transform of $e^{jt_0\omega}$ *sinc* $\left(\frac{\omega}{2\pi}\right)$.

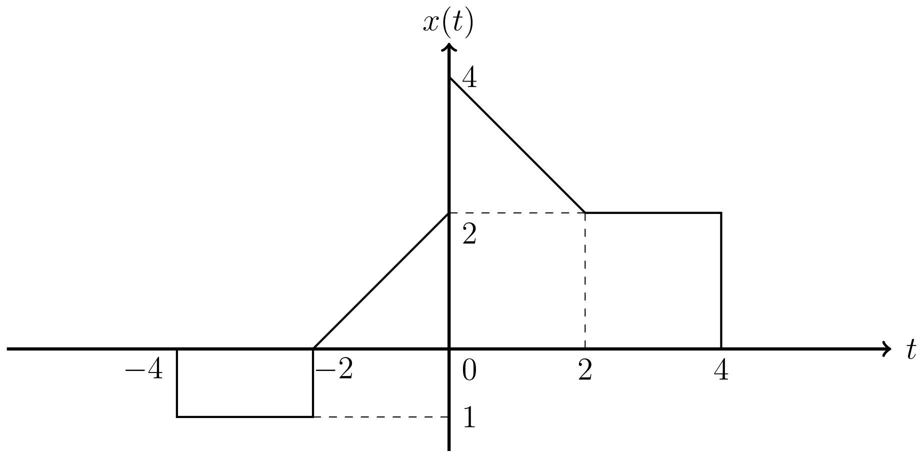
Solution: We can rewrite $X(\omega)$ as

$$\begin{aligned}
X(\omega) &= \frac{\left(e^{jb\omega} + e^{-jb\omega}\right)}{2}\frac{e^{ja\omega} - e^{-jb\omega}}{2j}\text{sinc}(\frac{\omega}{2\pi}) \\
&= \frac{1}{4j}\left[e^{j(a+b)\omega} + e^{j(a-b)\omega} - e^{j(-a+b)\omega} - e^{j(-a-b)\omega}\right]\text{sinc}(\frac{\omega}{2\pi}) \\
&= \frac{1}{4j}\left[e^{j(a+b)\omega} + e^{j(a-b)\omega} - e^{j(-a+b)\omega} - e^{j(-a-b)\omega}\right]\text{sinc}(\frac{\omega}{2\pi})
\end{aligned}$$

Using the time-shifting property and the fact that sinc$(\frac{\omega}{2\pi})$ is the Fourier transform of $\Pi(t)$, we get that

$$\boxed{x(t) = \frac{1}{4j}\Big[\Pi(t+a+b) + \Pi(t+a-b) - \Pi(t-a+b) - \Pi(t-a-b)\Big]}$$

3. (4 points) Find the Fourier transform of the signal depicted in the following figure.



Hint: You can express $x(t)$ as a linear combination of other signals for which you know the Fourier transform pair.

Solution: Note that, we can write $x(t)$ as

$$x(t) = 2\Pi\left(\frac{t-2}{4}\right) + 2\Lambda\left(\frac{t}{2}\right) - \Pi\left(\frac{t+3}{2}\right)$$

Then through the time-shifting and scaling properties of Fourier transform, we get that

$$\boxed{X(\omega) = 8\ \mathrm{sinc}\left(\frac{4\omega}{2\pi}\right) e^{-j2\omega} + 4\ \mathrm{sinc}^2\left(\frac{2\omega}{2\pi}\right) - 2\ \mathrm{sinc}\left(\frac{2\omega}{2\pi}\right) e^{j3\omega}}$$

Problem 2 (8 points) For the following questions, you do not need to do one to proceed with the next - you can use the statements of the previous questions as facts if you need them. **Furthermore, please answer the following questions without using Fourier Series or Fourier transform**.

1. (3 points) Prove the following property of the derivative of convolution, where $\star$ stands for convolution.

$$\frac{d}{dt}\Big(f(t) \star g(t)\Big) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

Hint: Recall, that the differentiator system, that takes as input a signal and outputs its derivative, is an LTI system. You can use this without proving it.

Solution:

Consider three LTI systems connected in cascade as follows:

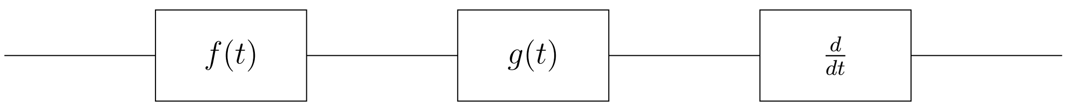


Figure 1: Response of $\delta(t)$ is $\frac{d}{dt}\Big(f(t) \star g(t)\Big)$.

Now, since we can reorder these LTI blocks, impulse responses of following two systems (Fig. 2 and Fig. 3) will also be same.

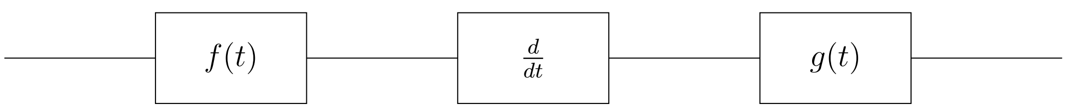


Figure 2: Response of $\delta(t)$ is $\left(\frac{d}{dt}f(t)\right) \star g(t)$.

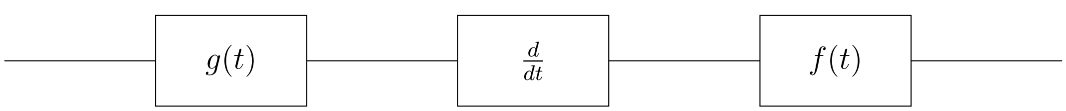


Figure 3: Response of $\delta(t)$ is $\left(\frac{d}{dt}g(t)\right) \star f(t) = f(t) \star \left(\frac{d}{dt}g(t)\right)$.

Thus,

$$\frac{d}{dt}\Big(f(t) \star g(t)\Big) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

2. (3 points) For the two signals $x_1(t) = \Pi(\frac{t}{2})$, $x_2(t) = e^{-5|t|}$, find the derivative of the convolution (in the time domain) $z(t) = x_1(t) \star x_2(t)$, that is, find $\frac{d}{dt}z(t)$.

Solution:

$$\begin{aligned}
\frac{d}{dt}\Big(x_1(t) \star x_2(t)\Big) &= \left(\frac{d}{dt}x_1(t)\right) \star x_2(t) \\
&= \left(\frac{d}{dt}\Pi\left(\frac{t}{2}\right)\right) \star e^{-5|t|} \\
&= \left(\frac{d}{dt}\left(u(t+1) - u(t-1)\right)\right) \star e^{-5|t|} \\
&= (\delta(t+1) - \delta(t-1)) \star e^{-5|t|} \\
&= e^{-5|t+1|} - e^{-5|t-1|}.
\end{aligned}$$

3. (2 points) Consider an LTI system, and assume that when the input is $4u(t-1)$ the output is $\cos^2(t)$. Find the impulse response (that is the response to $\delta(t)$) of this system?

   Hint: You can use the differentiation property in Question 1.

   Solution: Lets consider $h(t)$ is the impulse response. Then $4u(t-1) \star h(t) = \cos^2(t)$. Thus,

$$
\begin{aligned}
\frac{d}{dt} y(t) &= \frac{d}{dt}\Big(x(t) \star h(t)\Big) \\
\frac{d}{dt}\cos^2(t) &= \frac{d}{dt}\Big(4u(t-1) \star h(t)\Big) \\
-2\cos(t)\sin(t) &= \left(\frac{d}{dt}\left(4u(t-1)\right)\right) \star h(t) \\
-\sin(2t) &= 4\delta(t-1) \star h(t) \\
-\sin(2t) &= 4h(t-1).
\end{aligned}
$$

   Thus, $h(t) = -\frac{1}{4}\sin(2t+2)$.

Problem 3 (7 points) Consider a periodic signal $x(t)$, that has the power spectrum depicted on Fig. 4, where C_k is the coefficient of $e^{\frac{j2\pi kt}{T}}$ in the Fourier Series expansion of $x(t)$. Recall that in this plot, because $e^{\frac{j2\pi kt}{T}}$ has the frequency of $\frac{k}{T}$, we associate the magnitude square $|C_k|^2$ with the frequency $\frac{k}{T}$. The following questions are not related to each other.

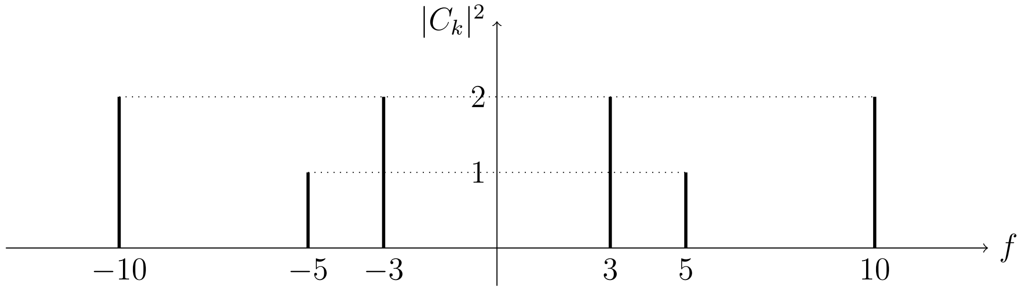


Figure 4: Power Spectrum of $x(t)$.

1. (2 points) Assume that $x(t)$ with power spectrum in Fig. 4 is real and even. Is there a unique $x(t)$ that has this power spectrum ? Explain why or why not.

   Solution: No, it will not be unique. As $x(t)$ and $-x(t)$ will have same power spectrum.

2. (2 points) Assume that $x(t)$ with power spectrum in Fig. 4 is the input to an LTI system. Is it possible that the power spectrum of $y(t)$ (the response to $x(t)$), is the one depicted in Fig. 5? Explain why or why not.

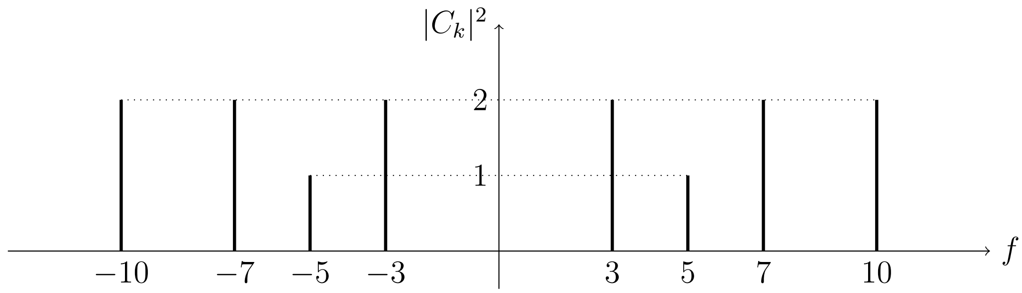


Figure 5: Power Spectrum of $y(t)$.

Solution: No, it can not be explained by an LTI System, as LTI systems can not generate new frequencies. There are frequency components at 7 and -7, which is not possible to have been generated by the LTI system there are no such components at $x(t)$.

3. (3 points) What is the fundamental period T for the signal $x(t)$ with power spectrum in Fig. 4? Explain your answer.

Solution:

Suppose coefficient at frequency f is D_f in $x(t)$, then we can write $x(t)$ as:

$$x(t) = D_{-10}e^{-j2\pi 10t} + D_{-5}e^{-j2\pi 5t} + D_{-3}e^{-j2\pi 3t} + D_3 e^{j2\pi 3t} + D_5 e^{j2\pi 5t} + D_{10}e^{j2\pi 10t}.$$

Each of these component have period $\frac{1}{10}, \frac{1}{5}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}$ and $\frac{1}{10}$ respectively. Thus the period of $x(t)$ is 1. And since each of these component are orthogonal, this is also the fundamental period.

1 Properties of Fourier Series

$x(t)$ and $y(t)$ are periodic signals of period T. ($\omega_0 = \frac{2\pi}{T}$)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt$$

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}, \quad d_k = \frac{1}{T}\int_T y(t) e^{-jk\omega_0 t}\, dt$$

Property	Signal	k^{th} Fourier coefficient
	$x(t)$ $y(t)$	c_k d_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha c_k + \beta d_k$
Time-Shifting	$x(t - t_0)$	$e^{-jk\omega_0 t_0} c_k$
Conjugation	$x^*(t)$	c^*_{-k}
Time-Reversal	$x(-t)$	c_{-k}
Time-Scaling	$x(\alpha t), \alpha > 0$ Period : $\frac{T}{\alpha}$	c_k
Conjugate-Symmetry	$x(t)$ is real	$c_k = c^*_{-k}$
Even-Odd Signals	$x(t)$ is real and even $x(t)$ is real and odd	c_k is real and even c_k is purely imaginary and odd

Parsevals Relation for Periodic Signals: $\frac{1}{T}\int\limits_T |x(t)|^2 dt = \sum\limits_{k=-\infty}^{\infty} |c_k|^2$

2 Fourier Transform Formulas

Fourier transform formulas (using ω):

Synthesis equation (Inverse Fourier Transform): $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

Analysis equation (Fourier Transform): $X(\omega) = \int\limits_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Fourier transform formulas (using f):

Synthesis equation (Inverse Fourier Transform): $x(t) = \int\limits_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

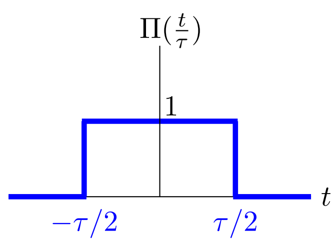
Analysis equation (Fourier Transform): $X(f) = \int\limits_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

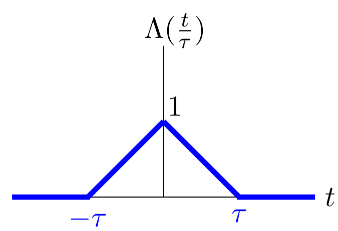
3 Fourier Transform Properties

Property	Signal	Fourier Transform
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$
Conjugate symmetry	$x(t)$ is real	$X^*(\omega) = X(-\omega)$
Conjugate anti-symmetry	$x(t)$ is purely imaginary	$X^*(\omega) = -X(-\omega)$
Even and real signal	$x(-t) = x(t)$	$\text{Im}\{X(\omega)\} = 0$
Odd and real signal	$x(-t) = -x(t)$	$\text{Re}\{X(\omega)\} = 0$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Time and frequency scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$
Differentiation in time	$\frac{d^n}{dt^n}[x(t)]$	$(j\omega)^n X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$

$$\text{Parseval's theorem: } \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

4 Fourier Transform pairs





$$\textbf{We define, sinc}(x) := \frac{\sin(\pi x)}{\pi x}$$

Name	Signal	Fourier Transform
Rectangular pulse	$x(t) = A\ \Pi(t/\tau)$	$X(\omega) = A\tau\ \text{sinc}(\frac{\omega\tau}{2\pi})$
Triangular pulse	$x(t) = A\ \Lambda(t/\tau)$	$X(\omega) = A\tau\ \text{sinc}^2(\frac{\omega\tau}{2\pi})$
Right-sided exponential	$x(t) = e^{-at}u(t)$	$X(\omega) = \frac{1}{a+j\omega}$
Two-sided exponential	$x(t) = e^{-a\lvert t\rvert}$	$X(\omega) = \frac{2a}{a^2+\omega^2}$
Unit impulse	$x(t) = \delta(t)$	$X(\omega) = 1$
Sinc function	$x(t) = \text{sinc}(t)$	$X(\omega) = \Pi(\frac{\omega}{2\pi})$
Constant-amplitude signal	$x(t) = 1,\ \text{all } t$	$X(\omega) = 2\pi\delta(\omega)$
Unit-step function	$x(t) = u(t)$	$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$