EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, November 15, 2017

NAME: [redacted] UID: [redacted]

This exam has 3 problems, for a total of 25 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.

Please, write your name and UID on the top of each loose sheet!

GOOD LUCK!

Problem	Points	Total Points
1	9	10
2	7.5	8
3	7	7
Total	23.5	25

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1 (10 points) The following questions are not related.

1. (3 points) Consider a system where the output $w(t)$ depends on the input $v(t)$ through the equation: $w(t) = \cos(v(t))$. Is this system time invariant? is it linear? is it stable? Explain why or why not.

Time invariant

Let $x_1(t) \mapsto y_1(t) = \cos(x_1(t))$

$x_1(t-t_0) \mapsto \cos(x_1(t-t_0))$

$y_1(t-t_0) = \cos(x_1(t-t_0))$ ✓ yes time invariant

Linear

$x_1(t) \mapsto y_1(t) = \cos(x_1(t))$

$x_2(t) \mapsto y_2(t) = \cos(x_2(t))$

$ax_1(t) + bx_2(t) \mapsto = \cos(ax_1(t) + bx_2(t))$

$ay_1(t) + by_2(t) = a\cos(x_1(t)) + b\cos(x_2(t))$ X Not linear

Stable

Let $|v(t)| \leq A$ for all t

$w(t) \leq 1$ for all t ✓ Stable

3

2. (3 points) Find the time domain representation for a signal with **Fourier Transform:**

$$X(\omega) = \cos(a\omega)\sin(b\omega)\text{sinc}\left(\frac{\omega}{2\pi}\right).$$

Hint: You can easily find the inverse Fourier Transform of $e^{jt_0\omega}\,sinc\left(\frac{\omega}{2\pi}\right)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(aw)\sin(bw)\text{sinc}\left(\frac{w}{2\pi}\right)e^{jwt}\,dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{jaw}+e^{-jaw}}{2}\right)\left(\frac{e^{jbw}-e^{-jbw}}{2j}\right)\text{sinc}\left(\frac{w}{2\pi}\right)e^{jwt}\,dw$$

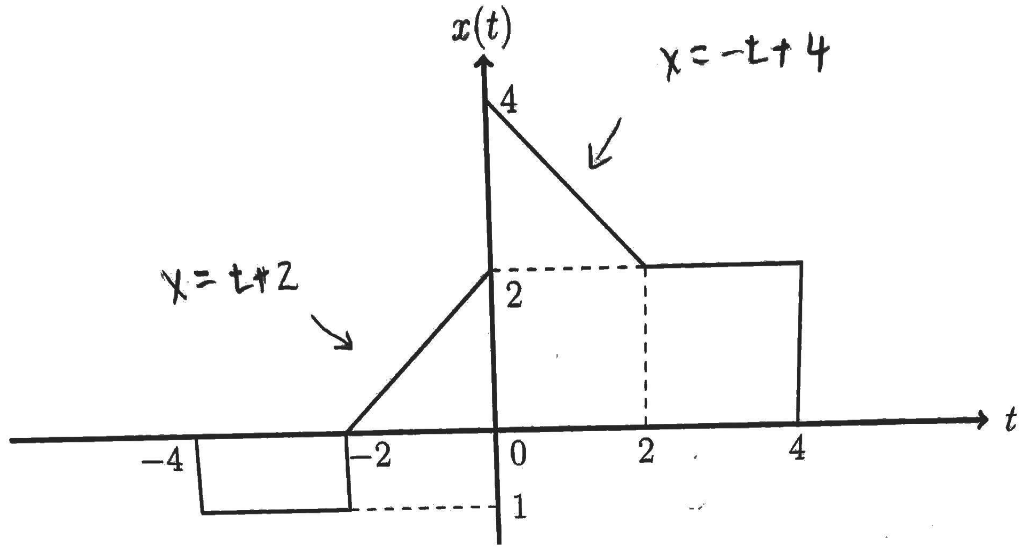
$$= \frac{1}{j8\pi}\int_{-\infty}^{\infty}\left(e^{jaw+jbw+jwt} + e^{-jaw+jbw+jwt} - e^{jaw-jbw+jwt} - e^{-jaw-jbw+jwt}\right)\text{smc}\left(\frac{w}{2\pi}\right)dw$$

Inverse transform of $\text{sinc}\left(\frac{w}{2\pi}\right) \Rightarrow \Pi(t)$ $\qquad e^{jt_0w}\text{sinc}\left(\frac{w}{2\pi}\right) \Rightarrow \Pi(t+t_0)$

$$= \frac{1}{4j}\left[\Pi(t+a+b) + \Pi(t-a+b) - \Pi(t+a-b) - \Pi(t-a-b)\right]$$

3

3. (4 points) Find the Fourier transform of the signal depicted in the following figure.



Hint: You can express $x(t)$ as a linear combination of other signals for which you know the Fourier transform pair.

$$X(w)=\int_{-\infty}^{\infty} x(t)e^{-jwt}\,dt$$

$$=\int_{-4}^{-2} -e^{-jwt}\,dt+\int_{-2}^{0}(t+2)e^{-jwt}\,dt+\int_{0}^{2}(-t+4)e^{-jwt}\,dt+\int_{2}^{4}2e^{-jwt}\,dt$$

$$=\frac{e^{-jwt}}{jw}\Big|_{-4}^{-2}+\frac{-2e^{-jwt}}{jw}\Big|_{-2}^{0}+\left(\frac{jt}{w}e^{-jwt}+\frac{e^{-jwt}}{w^2}\right)\Big|_{-2}^{0}+\frac{-4e^{-jwt}}{jw}\Big|_{0}^{2}$$

$$+\left(-\frac{jt}{w}e^{-jwt}-\frac{e^{-jwt}}{w^2}\right)\Big|_{0}^{2}+\frac{-2e^{-jwt}}{jw}\Big|_{2}^{4}$$

$$=\frac{1}{jw}\left(e^{2jw}-e^{4jw}-2+2e^{2jw}-4\cancel{e^{-2jw}}+4-2e^{-4jw}+2e^{-2jw}\right)+\frac{1}{w^2}+\frac{2j}{w}e^{2jw}-\frac{e^{2jw}}{w^2}$$

$$-\frac{2j}{w}e^{-2jw}-\frac{e^{-2jw}}{w^2}+\cancel{\frac{1}{w^2}}$$

$$=\frac{1}{jw}\left(3e^{2jw}-e^{4jw}+2-2e^{-2jw}-2e^{-4jw}\right)+\frac{2}{w^2}+\frac{2j}{w}\left(e^{2jw}-e^{-2jw}\right)-\frac{1}{w^2}\left(e^{2jw}+e^{-2jw}\right)$$

That did not work

$-\Pi\left(\frac{t+3}{2}\right)$ $\quad 2\Pi\left(\frac{t-2}{2}\right)$

$$x(t)=-\Pi\left(\frac{t}{2}+3\right)+2\Lambda\left(\frac{t}{2}\right)+2\Pi\left(\frac{t}{2}-2\right)$$

$$X(w)=-2\,\text{sinc}\left(\frac{w}{\pi}\right)e^{j3w}+\cancel{2}\,4\,\text{sinc}^2\left(\frac{w}{\pi}\right)+4\,\text{sinc}\left(\frac{w}{\pi}\right)e^{-2jw}$$

3

Problem 2 (8 points) For the following questions, you do not need to do one to proceed with the next - you can use the statements of the previous questions as facts if you need them. **Furthermore, please answer the following questions without using Fourier Series or Fourier transform.**

1. (3 points) Prove the following property of the derivative of convolution, where $\star$ stands for convolution.

$$\frac{d}{dt}\left(f(t) \star g(t)\right) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

$g(t) \mapsto g'(t)$

$g(t-t_0) \mapsto g'(t-t_0)$

Hint: Recall, that the differentiator system, that takes as input a signal and outputs its derivative, is an LTI system. You can use this without proving it.

$\frac{d}{dt}\left(f(t) * g(t)\right) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau\right)$

what is this?

$= \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \frac{d}{dt} = \int_{-\infty}^{\infty} f(\tau) g'(t-\tau) d\tau$

$= f(t) * \left(\frac{d}{dt} g(t)\right)$

2.5

$\frac{d}{dt}\left(f(t) * g(t)\right) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau\right)$

$= \int_{-\infty}^{\infty} g(\tau) f'(t-\tau) d\tau = g(t) * \left(\frac{d}{dt} f(t)\right) = \left(\frac{d}{dt} f(t)\right) * g(t)$

$\int g(\tau) \cdot \frac{d f(t-\tau)}{d(t-\tau)} \cdot d\tau$

2. **(3 points)** For the two signals $x_1(t) = \Pi(\frac{t}{2})$, $x_2(t) = e^{-5|t|}$, find the derivative of the convolution (in the time domain) $z(t) = x_1(t) \star x_2(t)$, that is, find $\frac{d}{dt}z(t)$.

$$\frac{d}{dt}z(t) = \frac{d}{dt}\left(x_1(t) * x_2(t)\right) = \frac{d}{dt}x_2(t) * x_1(t)$$

for $t<0$, $x_2(t) = e^{5t}$, $\frac{d}{dt}x_2(t) = 5e^{5t}$

for $t \geq 0$, $x_2(t) = e^{-5t}$, $\frac{d}{dt}x_2(t) = -5e^{-5t}$

$$\int_{-\infty}^{0} 5e^{5\tau}\,\Pi\left(\frac{\tau-t}{2}\right)d\tau + \int_0^\infty -5e^{-5\tau}\,\Pi\left(\frac{\tau-t}{2}\right)d\tau$$

$$= \int_{t-1}^{t+1 \text{ or } 0} 5e^{5\tau}\,d\tau + \int_{t-1 \text{ or } 0}^{t+1} -5e^{-5\tau}\,d\tau$$

$$= e^{5(t+1)} - e^{5(t-1)} - e^{-5(t+1)} + e^{-5(t-1)}$$

$$\therefore \frac{d}{dt}z(t) \begin{cases} e^{5(t+1)} - e^{5(t-1)}, & t<-1 \\ \cancel{2} - e^{5(t-1)} \cancel{} + e^{-5(t-1)}, & -1<t<1 \\ -e^{5(t+1)} + e^{-5(t-1)}, & t>1 \end{cases}$$

3

3. (2 points) Consider an LTI system, and assume that when the input is $4u(t-1)$ the output is $\cos^2(t)$. Find the impulse response (that is the response to $\delta(t)$) of this system?

Hint: You can use the differentiation property in Question 1.

$$\frac{d}{dt}\left(4u(t-1)\right) = 4\delta(t-1)$$

$$\frac{d}{dt}\left(4u(t-1) * h(t)\right) = -2\cos t \sin t = 4\delta(t-1) * h(t)$$

$$-2\cos t \sin t = 4h(t-1)$$

$$h(t) = -\frac{1}{2}\cos(t+1)\sin(t+1)$$

2

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Problem 3 (7 points) Consider a periodic signal $x(t)$, that has the power spectrum depicted on Fig. 1, where C_k is the coefficient of $e^{\frac{j2\pi kt}{T}}$ in the Fourier Series expansion of $x(t)$. Recall that in this plot, because $e^{\frac{j2\pi kt}{T}}$ has the frequency of $\frac{k}{T}$, we associate the magnitude square $|C_k|^2$ with the frequency $\frac{k}{T}$. The following questions are not related to each other.

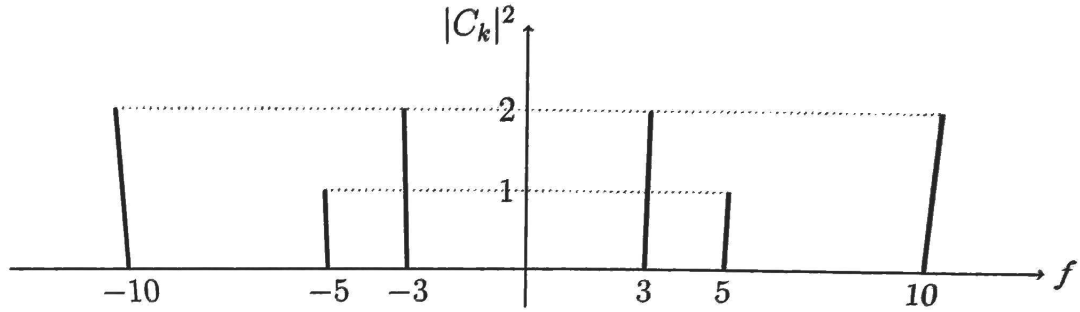

Figure 1: Power Spectrum of $x(t)$.

1. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is real and even. Is there a unique $x(t)$ that has this power spectrum ? Explain why or why not.

There is not a unique $x(t)$ that has this power spectrum because the power spectrum takes $|C_K|^2$, which means that even if we know that C_K is real, we don't know whether C_K is positive or negative. ✓ For example, for $f=3$, there can be an $x(t)$ that has a C_K of $-\sqrt{2}$, or an $x(t)$ that has a C_K of $\sqrt{2}$, and the resulting point at $f=3$ on the power spectrum looks the same.

2/2

2. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is the input to an *LTI* system. Is it possible that the power spectrum of $y(t)$ (the response to $x(t)$), is the one depicted in Fig. 2? Explain why or why not.

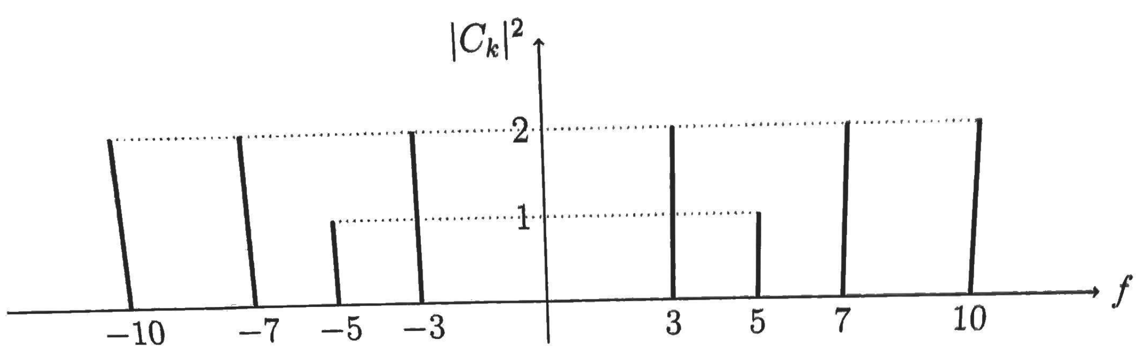

Figure 2: Power Spectrum of $y(t)$.

This is not possible because there are additional points at $f = -7$ and 7, which means that we are adding frequencies to the response of $x(t)$.

✓

2/2

3. (3 points) What is the fundamental period T for the signal $x(t)$ with power spectrum in Fig. 1? Explain your answer.

The fundamental period T of $x(t)$ must be one such that $\frac{k_1}{T_0} = 3$, $\frac{k_2}{T_0} = 5$, $\frac{k_3}{T_0} = 10$

where k_1, k_2, k_3 are integers. 3, 5, 10 have a common factor of 1

so $T_0 = 1$

✓