Total: 25 points

**EE102: Signals and Systems**

Midterm Exam

8:05 am - 9:35 am, November 15, 2017

**NAME:** [redacted] **UID:** [redacted]

This exam has 3 problems, for a total of 25 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
**Please, write your name and UID on the top of each loose sheet!**
**GOOD LUCK!**

| Problem | Points | Total Points |
|---|---|---|
| 1 | 8.5 | 10 |
| 2 | 8 | 8 |
| 3 | 5 | 7 |
| Total | 21.5 | 25 |

**Extra Pages:** ________

To fill in, in case extra sheets are used apart from what is provided.

**Note: Answers without justification will not be awarded any marks.**

Problem 1 (10 points) The following questions are not related.

1. (3 points) Consider a system where the output $w(t)$ depends on the input $v(t)$ through the equation: $w(t) = \cos(v(t))$. Is this system time invariant? is it linear? is it stable? Explain why or why not.

Time invariant:

$x(t) \mapsto y(t) = \cos(x(t))$

let $x_1(t) = x(t-t_0)$

$x_1(t) \mapsto y_1(t) = \cos(x_1(t)) = \cos(x(t-t_0))$

$x(t-t_0) \mapsto y(t-t_0) = \cos(x(t-t_0))$

equivalent

$\therefore$ system is $\boxed{\text{time invariant}}$

Linear:

$x_1(t) \mapsto y_1(t) = \cos(x_1(t))$

$x_2(t) \mapsto y_2(t) = \cos(x_2(t))$

let $x_3(t) = a x_1(t) + b x_2(t)$

$x_3(t) \mapsto y_3(t) = \cos[a x_1(t) + b \cdot x_2(t)]$

$\neq a y_1(t) + b y_2(t)$

$\therefore$ system is $\boxed{\text{not linear}}$

3

Stable:

If $|x(t)| \leq A$, then $y(t) = \underbrace{\cos(x(t))}$

cosine function bounded by $\pm 1$

Because a bounded input yields a bounded output, the system is $\boxed{\text{stable}}$

2. (3 points) Find the time domain representation for a signal with Fourier Transform:

$$X(\omega) = \cos(a\omega)\sin(b\omega)\text{sinc}\left(\frac{\omega}{2\pi}\right).$$

*Hint: You can easily find the inverse Fourier Transform of* $e^{jt_0\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$.

$X(\omega) = \cos(a\omega)\sin(b\omega)\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$

$\cos(a\omega) = \frac{1}{2}\left(e^{ia\omega} + e^{-ia\omega}\right) \quad \sin(b\omega) = \frac{1}{2j}\left(e^{ib\omega} - e^{-ib\omega}\right)$

$X(\omega) = \frac{1}{4}\left(e^{ia\omega} + e^{-ia\omega}\right)\left(e^{ib\omega} - e^{-ib\omega}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) \qquad -ia\omega - ib\omega = i(-a-b)\omega$

$= \frac{1}{4}\left[e^{i(a+b)\omega} + e^{i(b-a)\omega} - e^{i(a-b)\omega} - e^{i(-a-b)\omega}\right]\text{sinc}\left(\frac{\omega}{2\pi}\right)$

let $X_1(\omega) = e^{i(a+b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$, $X_2(\omega) = \frac{1}{4}e^{i(b-a)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$, $X_3(\omega) = e^{i(a-b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$.

$X_4(\omega) = e^{i(-a-b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right)$ $\qquad$ * $\text{sinc}\left(\frac{\omega}{2\pi}\right) \longleftrightarrow \Pi(t)$

By linearity, $X(t) = \frac{1}{4}\left(X_1(t) + X_2(t) - X_3(t) - X_4(t)\right)$

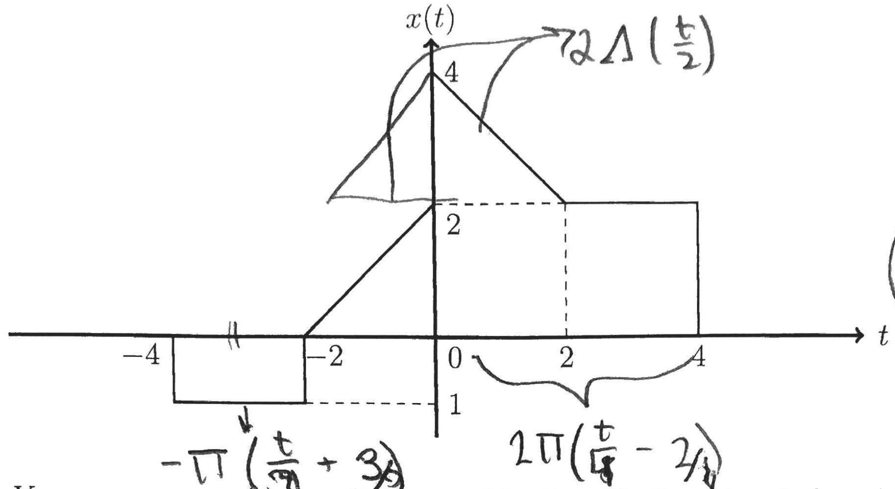
$e^{i(a+b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right) \rightarrow$ time shift $\longleftrightarrow \Pi(t+a+b)$. $\quad e^{i(a-b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right) \longleftrightarrow \Pi(t+a-b)$

$e^{i(b-a)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right) \longleftrightarrow \Pi(t+b-a) \quad e^{i(-a-b)\omega}\,\text{sinc}\left(\frac{\omega}{2\pi}\right) \longleftrightarrow \Pi(t-a-b)$

$\therefore \boxed{X(t) = \frac{1}{4j}\left[\Pi(t+a+b) + \Pi(t+b-a) - \Pi(t+a-b) - \Pi(t-a-b)\right]}$

3

3. (4 points) Find the Fourier transform of the signal depicted in the following figure.




2.5

*Hint: You can express $x(t)$ as a linear combination of other signals for which you know the Fourier transform pair.*

$$x(t) = \underbrace{2\Pi\left(\frac{t}{8}-2\right)}_{x_1(t)} + \underbrace{2\Lambda\left(\frac{t}{2}\right)}_{x_2(t)} - \underbrace{\Pi\left(\frac{t}{4}+3\right)}_{x_3(t)}$$

By linearity, $X(\omega) = X_1(\omega) + X_2(\omega) - X_3(\omega)$

$\Pi(at-b) \longleftrightarrow \frac{1}{|a|} e^{-\omega b}\,\mathrm{sinc}\left(\frac{\omega}{2\pi a}\right)$

$$x_1(t) = 2\Pi\left(\frac{t}{8}-2\right) \longleftrightarrow \frac{2}{|1/8|} e^{-i2\omega}\,\mathrm{sinc}\left(\frac{8\omega}{2\pi}\right) = 16e^{-i2\omega}\,\mathrm{sinc}\left(\frac{4\omega}{\pi}\right)$$

$a = 1/8 \quad b = 2$

$$x_2(t) = 2\Lambda\left(\frac{t}{2}\right) \longleftrightarrow 2\cdot\frac{1}{|1/2|}\,\mathrm{sinc}^2\left(\frac{2\omega}{2\pi}\right) = 4\,\mathrm{sinc}^2\left(\frac{\omega}{\pi}\right)$$

$$x_3(t) = \Pi\left(\frac{t}{4}+3\right) \longleftrightarrow \frac{1}{|1/4|} e^{i3\omega}\,\mathrm{sinc}\left(\frac{4\omega}{2\pi}\right) = 4e^{i3\omega}\,\mathrm{sinc}\left(\frac{2\omega}{\pi}\right)$$

$a = 1/4 \quad b = -3$

$$\therefore \boxed{X(\omega) = 16e^{-i2\omega}\,\mathrm{sinc}\left(\frac{4\omega}{\pi}\right) + 4\,\mathrm{sinc}^2\left(\frac{\omega}{\pi}\right) - 4e^{i3\omega}\,\mathrm{sinc}\left(\frac{2\omega}{\pi}\right)}$$

Problem 2 (8 points) For the following questions, you do not need to do one to proceed with the next - you can use the statements of the previous questions as facts if you need them. **Furthermore, please answer the following questions without using Fourier Series or Fourier transform.**

1. (3 points) Prove the following property of the derivative of convolution, where $\star$ stands for convolution.

$$\frac{d}{dt}\Big(f(t) \star g(t)\Big) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

*Hint: Recall, that the differentiator system, that takes as input a signal and outputs its derivative, is an LTI system. You can use this without proving it.*

Treating differentiation with respect to time as an LTI system: let $h(t)$ = impulse response to the differentiator system →

$$\frac{d}{dt}\left(f(t) * g(t)\right) = h(t) * \left[f(t) * g(t)\right]$$

$$= \left[h(t) * f(t)\right] * g(t) = f(t) * \left[h(t) * g(t)\right]$$

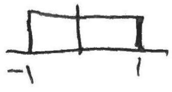
$$= \boxed{\left(\frac{d}{dt}f(t)\right) * g(t)} \qquad = \boxed{f(t) * \left(\frac{d}{dt}g(t)\right)}$$

3

$$\therefore \frac{d}{dt}\left(f(t) * g(t)\right) = \left(\frac{d}{dt}f(t)\right) * g(t) = f(t) * \left(\frac{d}{dt}g(t)\right)$$

2. (3 points) For the two signals $x_1(t) = \Pi(\frac{t}{2})$, $x_2(t) = e^{-5|t|}$, find the derivative of the convolution (in the time domain) $z(t) = x_1(t) \star x_2(t)$, that is, find $\frac{d}{dt}z(t)$.

$$\frac{d}{dt}\left[x_1(t) * x_2(t)\right] = \left(\frac{d}{dt}x_1(t)\right) * x_2(t) = x_1(t) * \frac{d}{dt}x_2(t)$$

$\Pi\left(\frac{t}{2}\right) =$ 


$\Pi\left(\frac{t}{2}\right) = u(t+1)\cdot u(1-t)$

$$\therefore \frac{d}{dt}\left(\Pi\left(\frac{t}{2}\right)\right) = \frac{d}{dt}\left(u(t+1)\cdot u(1-t)\right) = \delta(t+1)u(1-t) + \delta(1-t)u(t+1)$$

$$= \delta(t+1)u(2) + \delta(1-t)u(2)$$

$$= \delta(t+1) + \delta(1-t)$$

$$= \delta(t+1) + \delta(t-1)$$

$$\frac{d}{dt}z(t) = \int_{-\infty}^{\infty}\left[\delta(\tau+1) + \delta(\tau-1)\right]\cdot e^{-5|(t-\tau)|}\,d\tau$$

$$= \int_{-\infty}^{\infty}\delta(\tau+1)e^{-5|t-\tau|}\,d\tau + \int_{-\infty}^{\infty}\delta(\tau-1)e^{-5|(t-\tau)|}\,d\tau$$

$$= \boxed{e^{-5|t+1|} + e^{-5|t-1|}}$$

3

3. (2 points) Consider an LTI system, and assume that when the input is $4u(t-1)$ the output is $\cos^2(t)$. Find the impulse response (that is the response to $\delta(t)$) of this system?

*Hint: You can use the differentiation property in Question 1.*

$$4u(t-1) * h(t) = \cos^2(t)$$

$$\frac{d}{dt} 4u(t-1) = 4\delta(t-1)$$

$$\frac{d}{dt}\left(\cos^2(t)\right) = \frac{d}{dt}\left[4u(t-1)\right] * h(t)$$

$$-2\cos t \sin t = 4\delta(t-1) * h(t)$$

$$-2\cos t \sin t = 4h(t-1)$$

$$-\frac{\cos t \sin t}{2} = h(t-1)$$

$$\boxed{h(t) = -\frac{\cos(t+1)\sin(t+1)}{2}}$$

2

5/7

Problem 3 (7 points) Consider a periodic signal $x(t)$, that has the power spectrum depicted on Fig. 1, where $C_k$ is the coefficient of $e^{\frac{j2\pi kt}{T}}$ in the Fourier Series expansion of $x(t)$. Recall that in this plot, because $e^{\frac{j2\pi kt}{T}}$ has the frequency of $\frac{k}{T}$, we associate the magnitude square $|C_k|^2$ with the frequency $\frac{k}{T}$. The following questions are not related to each other.

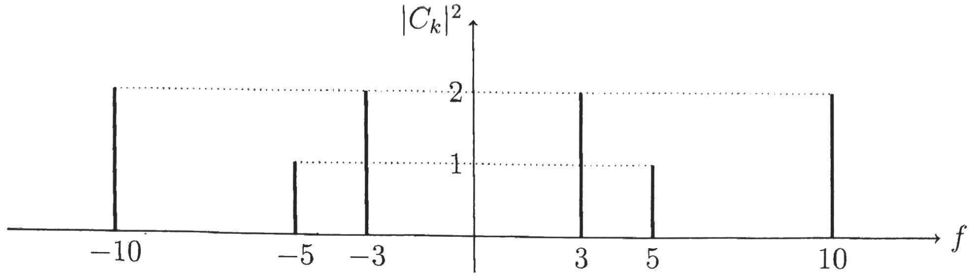



Figure 1: Power Spectrum of $x(t)$.

1. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is real and even. Is there a unique $x(t)$ that has this power spectrum ? Explain why or why not.

Because $x(t)$ is real and even, we know that its fourier series coefficients are real and even. What we see plotted on the power spectrum is the magnitude of the coefficients squared, which in this case is just the square of the coefficients because they are real. However, we do not know which coefficients were negative and which were positive from the power spectrum b/c the square makes everything positive. Therefore, a signal $y(t)$, for example, with the same fourier coefficients but negative, would have the same power spectrum.

∴ $\boxed{\text{It is not unique}}$

2/2

2. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is the input to an LTI system. Is it possible that the power spectrum of $y(t)$ (the response to $x(t)$), is the one depicted in Fig. 2? Explain why or why not.

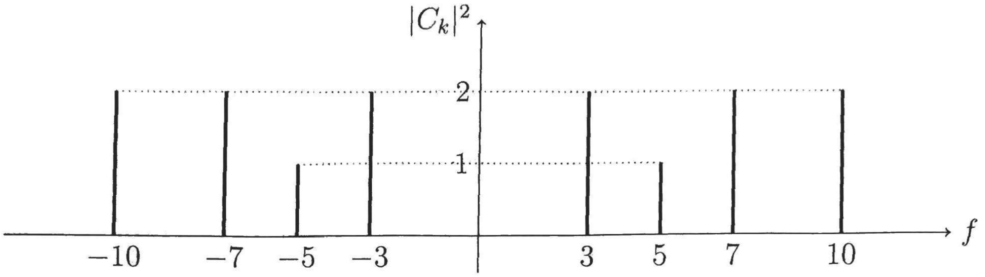



Figure 2: Power Spectrum of $y(t)$.

It is not possible for the above power spectrum to be the response of x(t) to an LTI system. We know that LTI systems can only attenuate or amplify existing frequencies power of certain frequencies, but it cannot add power to frequencies. y(t) has power at f=7 where x(t) doesn't, so it cannot be the response of x(t) to an LTI system

2/2

3. (3 points) What is the fundamental period $T$ for the signal $x(t)$ with power spectrum in Fig. 1? Explain your answer.

We know that a periodic signal can only have power at the harmonics of its natural frequency ($\frac{1}{T}$).
Because we have power at frequencies of $\frac{3}{T}$, $\frac{5}{T}$, and $\frac{10}{T}$, we know that the natural frequency must be the largest frequency which takes these harmonics,

LCM

$\therefore T =$ greatest common factor of $\frac{1}{3}, \frac{1}{5}$, and $\frac{1}{10}$

$\boxed{T = \frac{1}{30}}$ ← can not work

30 x integer = 5 ?

1/3