Total: 25 points

EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, November 15, 2017

NAME: [redacted] **UID:** [redacted]

This exam has 3 problems, for a total of 25 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	9.5	10
2	5	8
3	5.5	7
Total	20	25

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: **Answers without justification will not be awarded any marks.**

Problem 1 (10 points) The following questions are not related.

1. (3 points) Consider a system where the output $w(t)$ depends on the input $v(t)$ through the equation: $w(t) = \cos(v(t))$. Is this system time invariant? is it linear? is it stable? Explain why or why not.

a) Time invariant

$v_1(t) \rightarrow w_1(t) = \cos(v_1(t))$

$w_1(t-t_0) = \cos(v_1(t-t_0))$

$v_2(t) = v_1(t-t_0) \rightarrow w_2(t)$] = ∴ time invariant

$w_2(t) = \cos(v_1(t-t_0))$

1

b) $w(t) = \cos(v(t))$

$v_1(t) \rightarrow w_1(t) = \cos(v_1(t))$

$v_2(t) \rightarrow w_2(t) = \cos(v_2(t))$ 1

$v_3(t) \rightarrow av_1(t) + bv_2(t) \rightarrow w_3(t)$

$w_3(t) = \cos(v_3(t)) = \cos(av_1(t) + bv_2(t)) \neq a\cos(v_1(t)) + b\cos(v_2(t))$

$\neq aw_1(t) + bw_2(t)$

∴ the system is not linear

c) If v(t) is bounded, let $|v(t)| < \pi$

then w(t) is also bounded ∴ the system is stable

0.5

can be great

what if

$(2\pi > v(t) > \pi)$??

2. (3 points) Find the time domain representation for a signal with Fourier Transform:

$$X(\omega) = \cos(a\omega)\sin(b\omega)\text{sinc}\left(\frac{\omega}{2\pi}\right).$$

Hint: You can easily find the inverse Fourier Transform of $e^{jt_0\omega}$ *sinc* $\left(\frac{\omega}{2\pi}\right)$.

$X(w) = \frac{1}{2}(e^{jaw} + e^{-jaw}) \cdot \frac{1}{2j}(e^{jbw} - e^{-jbw}) \text{ sinc}\left(\frac{w}{2\pi}\right)$

$= \frac{1}{4j}\left(e^{jw(a+b)} - e^{-jw(b-a)} + e^{-jw(a-b)} - e^{-jw(a+b)}\right) \text{sinc}\left(\frac{w}{2\pi}\right)$

$X_1(w) = \text{sinc}\left(\frac{w}{2\pi}\right) \rightarrow \Pi(t)$

$X_2(w) = e^{jw(a+b)} X_1(w) \rightarrow \Pi(t+(a+b))$

$X_3(w) = -e^{-jw(b-a)} \text{sinc}\left(\frac{w}{2\pi}\right) \rightarrow -\Pi(t-(b-a)) = \Pi(t+a+b)$

$X_4(w) = e^{-jw(a-b)} \text{sinc}\left(\frac{w}{2\pi}\right) \rightarrow \Pi(t-(a-b)) = \Pi(t+a+b)$

$X_5(w) = e^{-jw(a+b)} \text{sinc}\left(\frac{w}{2\pi}\right) \rightarrow \Pi(t-(a+b)) = \Pi(t-a-b)$

So $X(t) = \frac{1}{4j}\left(\Pi(t+a+b) - \Pi(t+a+b) + \Pi(t+a+b) + \Pi(t-a-b)\right)$

$X(t) = \frac{1}{4j}\left(\Pi(t+(a+b)) + \Pi(t-(a+b))\right)$

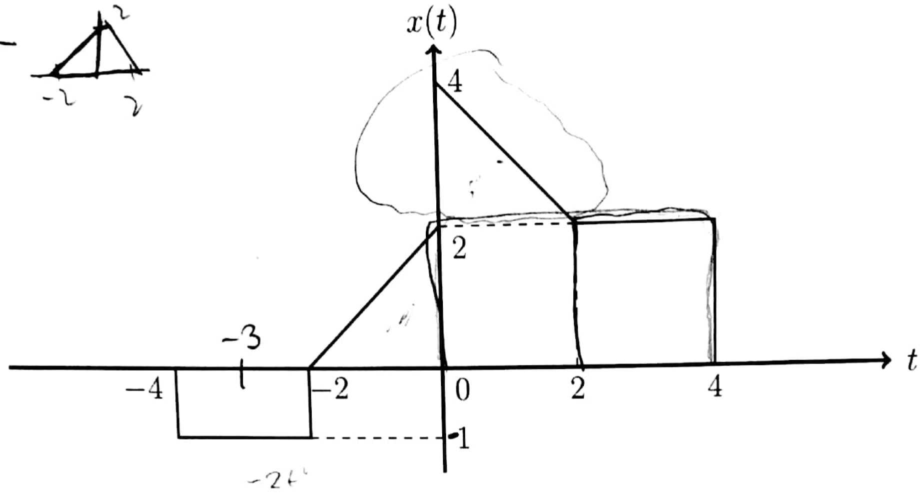
+ − Π

(3)

3. (4 points) Find the Fourier transform of the signal depicted in the following figure.

$\Lambda(t) =$

$2\Lambda(\frac{t}{2})$



Hint: You can express $x(t)$ as a linear combination of other signals for which you know the Fourier transform pair.

$x(t) = -\Pi\left(\frac{t+3}{2}\right) + \Lambda\left(\frac{t}{2}\right) + 2\Pi\left(\frac{t-2}{4}\right) + \Lambda\left(\frac{t}{2}\right)$

- $\Pi(t) \to \text{sinc}\left(\frac{w}{2\pi}\right)$

  $\Pi\left(t+\frac{3}{2}\right) \to e^{jw3/2}\,\text{sinc}\left(\frac{w}{2\pi}\right)$

  $-\Pi\left(\frac{t}{2}+\frac{3}{2}\right) \to -2e^{j3w}\,\text{sinc}\left(\frac{w}{\pi}\right)$

$\Lambda(t) \to \text{sinc}^2\left(\frac{w}{2\pi}\right)$

$2\Lambda\left(\frac{t}{2}\right) \to 2\cdot 2\cdot \text{sinc}^2\left(\frac{w}{\pi}\right)$

$= 4\,\text{sinc}^2\left(\frac{w}{\pi}\right)$

- $\Pi(t) \to \text{sinc}\left(\frac{w}{2\pi}\right)$

  $\Pi\left(t-\frac{1}{2}\right) \to e^{-jw/2}\,\text{sinc}\left(\frac{w}{2\pi}\right)$

  $\Pi\left(\frac{1}{4}t-\frac{1}{2}\right) \to 4e^{-j2w}\,\text{sinc}\left(\frac{2w}{\pi}\right)$

  $2\Pi\left(\frac{1}{4}t-\frac{1}{2}\right) \to 8e^{-j2w}\,\text{sinc}\left(\frac{2w}{\pi}\right)$

(4)

So overall $X(w) = -2e^{j3w}\,\text{sinc}\left(\frac{w}{\pi}\right) + 4\,\text{sin}^2\left(\frac{w}{\pi}\right) + 8e^{-j2w}\,\text{sinc}\left(\frac{2w}{\pi}\right)$

Problem 2 (8 points) For the following questions, you do not need to do one to proceed with the next - you can use the statements of the previous questions as facts if you need them. **Furthermore, please answer the following questions without using Fourier Series or Fourier transform.**

1. (3 points) Prove the following property of the derivative of convolution, where $\star$ stands for convolution.

$$\frac{d}{dt}\Big(f(t) \star g(t)\Big) = \left(\frac{d}{dt}f(t)\right) \star g(t) = f(t) \star \left(\frac{d}{dt}g(t)\right).$$

Hint: Recall, that the differentiator system, that takes as input a signal and outputs its derivative, is an LTI system. You can use this without proving it.

$\frac{d}{dt}x(t) \rightarrow$ LTI.

$f(t) \star g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$

$\frac{d}{dt}(f(t) \star g(t)) = \frac{d}{dt}\left(\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau\right)$

$\delta(t) \rightarrow$ [g(t)] $\rightarrow h(t)$

$f(t) \rightarrow$ [g(t)] $\rightarrow y(t)$

let $y(t) = f(t) \star g(t)$.

$h(t) = \delta(t) \star g(t) = g(t)$.

$\frac{d}{dt}(h(t)) = \frac{d}{dt}(g(t))$

$\frac{d}{dt}(h(t)) = \frac{d}{dt}(g(t)) \star \delta(t)$. ⇓ why??

so $\frac{d}{dt}(y(t)) = \frac{d}{dt}(g(t)) \star f(t)$ ✓

1.5

Now let $w(t) = g(t) \star f(t)$

$h(t) = \delta(t) \star f(t) = f(t)$.

$\frac{d}{dt}(h(t)) = \frac{d}{dt}(f(t)) = \frac{d}{dt}(f(t)) \star \delta(t)$ ⇓ why?

so $\frac{d}{dt}w(t) = \frac{d}{dt}(f(t)) \star g(t)$ ✓

Therefore since $\frac{d}{dt}y(t) = \frac{d}{dt}w(t)$ then $\frac{d}{dt}(f(t) \star g(t)) = \frac{d}{dt}(g(t)) \star f(t) = \frac{d}{dt}(f(t)) \star g(t)$

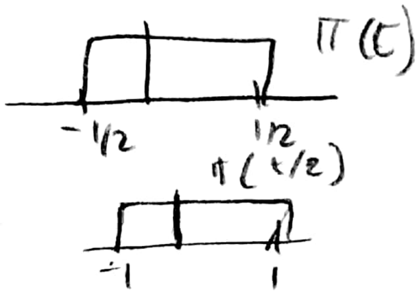
2. (3 points) For the two signals $x_1(t) = \Pi(\frac{t}{2})$, $x_2(t) = e^{-5|t|}$, find the derivative of the convolution (in the time domain) $z(t) = x_1(t) \star x_2(t)$, that is, find $\frac{d}{dt}z(t)$.

$x_1(t) * x_2(t) = \Pi\left(\frac{t}{2}\right) * e^{-5|t|}$

$= \int_{-\infty}^{\infty} \Pi\left(\frac{\tau}{2}\right) \cdot e^{-5|t-\tau|} \, d\tau$

$= \int_{-1}^{1} e^{-5|t-\tau|} \, d\tau$

$= \left. \frac{e^{-5|t-\tau|}}{-5} \right|_{-1}^{1}$ X



$\int e^{-|t|} \, dt$ is not $\frac{e^{-|t|}}{-1}$

$z(t) = \frac{e^{-5|t-1|}}{-5} + \frac{e^{-5|t+1|}}{5}$

$\frac{dz(t)}{dt} = e^{|t-1|} - e^{|t+1|}$ X

1.5

$(\cos t)^2$

3. (2 points) Consider an LTI system, and assume that when the input is $4u(t-1)$ the output is $\cos^2(t)$. Find the impulse response (that is the response to $\delta(t)$, of this system?

Hint: You can use the differentiation property in Question 1.

$4u(t-1) \rightarrow$ [LTI, $h(t)$] $\rightarrow \cos^2(t)$

$\delta(t) \rightarrow$ [$h(t)$] $\rightarrow h(t)$

$\cos^2 t = 4u(t-1) * h(t) = y(t)$

$\frac{d}{dt} y(t) = \left[\frac{d}{dt} 4u(t-1)\right] * h(t)$

$\frac{d}{dt} y(t) = 4\delta(t-1)$

$y(t) = \cos^2 t$

let $x(t) = 4u(t-1) \longmapsto \cos^2 t$

$4\delta(t-1) \longrightarrow h(t)$

$-2\cos t \sin t = 4\delta(t-1) * h(t)$

$-2\cos t \sin t = 4h(t-1)$ $\quad \sin(2t)$

$h(t-1) = -\frac{1}{2}\cos t \sin t = -\frac{1}{4}\cos 2t$

$h(t) = -\frac{1}{4}\cos(2(t+1))$

2

$\delta(t) \rightarrow$ [$h(t)$] $\rightarrow y(t)$

Impulse response $= y(t) = \delta(t) * h(t)$

$= \delta(t) * -\frac{1}{4}\cos(2t+2)$

8

$= -\frac{1}{4}\cos(2t+2)$

Note: Ans...

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Problem 3 (7 points) Consider a periodic signal $x(t)$, that has the power spectrum depicted on Fig. 1, where C_k is the coefficient of $e^{\frac{j2\pi kt}{T}}$ in the Fourier Series expansion of $x(t)$. Recall that in this plot, because $e^{\frac{j2\pi kt}{T}}$ has the frequency of $\frac{k}{T}$, we associate the magnitude square $|C_k|^2$ with the frequency $\frac{k}{T}$. The following questions are not related to each other.

$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{k2\pi t}{T}}$

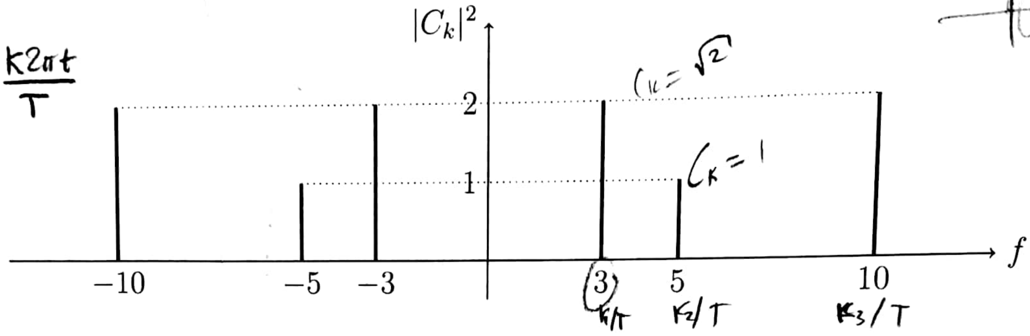


Figure 1: Power Spectrum of $x(t)$.

$3 = \frac{k}{T}$

$\frac{T}{k} = \frac{1}{3}$

1. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is real and even. Is there a unique $x(t)$ that has this power spectrum ? Explain why or why not.

If x(t) is real and even then C_k is also real and even $C_k = C_{-k} = C^*_{-k}$. For example, x(t) can be cos(t) since it is both real, even and periodic. There can be many signals to satisfy this condition so no, x(t) is not unique.

no, it cannot!

0.5/2

2. (2 points) Assume that $x(t)$ with power spectrum in Fig. 1 is the input to an LTI system. Is it possible that the power spectrum of $y(t)$ (the response to $x(t)$), is the one depicted in Fig. 2? Explain why or why not.

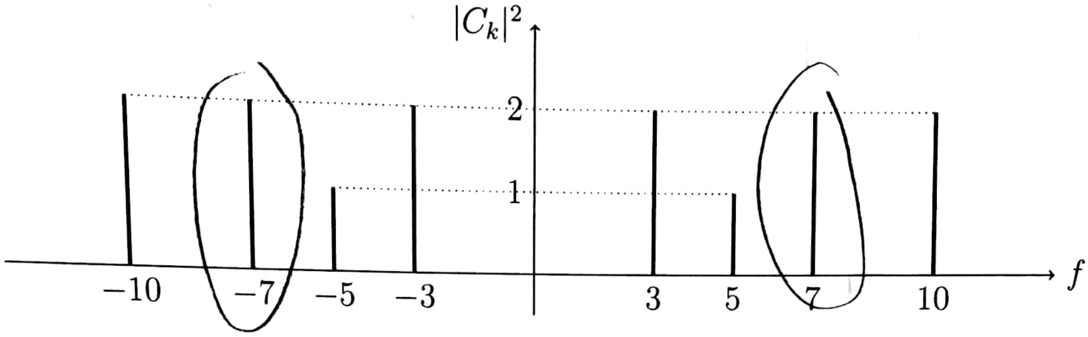
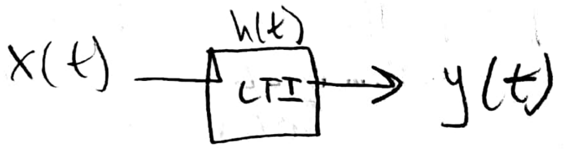


Figure 2: Power Spectrum of $y(t)$.



$y(t) = x(t) * h(t)$

$y(t) = e^{j\frac{2\pi k t}{T}} \int_{-\infty}^{\infty} h(\tau) e^{-j\frac{2\pi k \tau}{T}} d\tau$ H_K

H_K depends on K

$y(t) = H_K e^{j\frac{2\pi k t}{T}}$

$y(t) = \sum_{k=-\infty}^{\infty} C_K H_K e^{j2\pi k t/T}$

The power spectrum of $y(t)$ will depend on $|C_K H_K|^2$ rather than $|C_K|^2$ from $x(t)$. Since there's an extra H_K being squared then the power spectrum of $y(t)$ will be multiplied by a magnitude of $|H_K|^2$, therefore since the power spectrum depicted in Fig. 2 is similar to Fig. 1 then it cannot be the power spectrum of $y(t)$.

2/2

* LTI systems cannot generate new frequencies

3. (3 points) What is the fundamental period T for the signal $x(t)$ with power spectrum in Fig. 1? Explain your answer.

- $|C_3|^2 = 2 = |C_{-3}|^2$
- $|C_{10}|^2 = 2 = |C_{-10}|^2$

$3 = \frac{k}{T}$ $\quad 5 = \frac{k}{T}$

$3T = K$ $\quad 5 \neq \frac{3T}{T}$

$3 = \frac{k_1}{T}$ $\quad 5 = \frac{k_2}{T}$

$k_1 = 3$ $\quad k_2 = 5$

The fundamental period T is 1 — is the smallest value that satisfies the conditions

$3 = \frac{k_1}{T}$ $\quad 5 = \frac{k_2}{T}$ $\quad 10 = \frac{k_3}{T}$

$3T = k_1$ $\quad 5T = k_2$

$\frac{3k_2}{5} = k_1$ $\quad T = \frac{k_2}{5}$

$\frac{k_1}{k_2} = \frac{3}{5}$

So $k_1 = 3$ & $k_2 = 5$ then $T = 1$

~~$c_k = \frac{1}{T}\int_0^T x(t) e^{-jk\frac{2\pi}{T}} dt$~~

3/3