Total: 15 points

EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, October 23, 2017

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This exam has 3 problems, for a total of 15 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1	4	5
2	4.5	5
3	5	5
Total	13.5	15

Extra Pages: ________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

8

Problem 1(5 points) The following questions are not related to each other.

1. In the following expressions, $*$ stands for the convolution operation, $u(t)$ is the unit step function and $\delta(t)$ is the delta function.

   (a) Simplify $\frac{du(t)}{dt} * (te^{-6|t+1|}\delta(t-3))$.

   (b) Calculate $\int_{-\infty}^{\infty} f(t)g(t)\,dt$ with $f(t) = u(t+3)$ and $g(t) = u(-t+5)$.

   (c) Calculate $y(0)$, where $y(t) = x_1(t) * x_2(t)$, $x_1(t) = u(t-3) - u(t-1)$ and $x_2(t) = \delta(-2-t)$.

2. Consider the signal

$$x(t) = u(t-3)$$

and the signal

$$y(t) = \begin{cases} x(t) & \text{if} \quad t \geq 0.5 \\ -x(-t) & \text{otherwise} \end{cases}.$$

Is the signal $y(t)$ even or odd or neither? Justify your answer.

1) a- $\frac{du(t)}{dt} * (te^{-6|t+1|}\delta(t-3)) = \underbrace{\delta(t) * (te^{-6|t+1|}\delta(t-3))}_{\frac{du(t)}{dt} = \delta(t)} = \underbrace{(te^{-6|t+1|}\delta(t-3)) * \delta(t)}_{\text{commutativity of convolution}}$

$= \underbrace{te^{-6|t+1|}\delta(t-3)}_{\text{identity property of convolution w/ } \delta(t)} = \underbrace{(3)e^{-6|(3)+1|}\delta(t-3) = 3e^{-24}\delta(t-3)}_{\text{sifting property of the delta function}} = \boxed{\frac{3}{e^{24}}\delta(t-3)}$

1

b- $f(t) = u(t+3) = \begin{cases} 1 & -3 \leq t \\ 0 & t < -3 \end{cases}$

$g(t) = u(-t+5) = \begin{cases} 1 & 0 \leq (-t+5) \\ 0 & (-t+5) < 0 \end{cases} = \begin{cases} 1 & -5 \leq -t \\ 0 & -t < -5 \end{cases} = \begin{cases} 1 & t \leq 5 \\ 0 & 5 < t \end{cases}$

$\therefore f(t)\,g(t) = \begin{cases} 0 & -\infty < t < -3 \\ 1 & -3 \leq t \leq 5 \\ 0 & 5 < t < \infty \end{cases}$ $\therefore \int_{-\infty}^{\infty} f(t)g(t)\,dt = \int_{-\infty}^{-3} f(t)g(t)\,dt + \int_{-3}^{5} f(t)g(t)\,dt$

$+ \int_{5}^{\infty} f(t)g(t)\,dt = \int_{-3}^{5} f(t)g(t)\,dt = \int_{-3}^{5} dt = 5-(-3) = \boxed{8}$

1

c- $y(t) = \underbrace{[u(t-3) - u(t-1)] * \delta(-2-t)}_{\text{delay for convolution by } \delta} = [u(t-3) - u(t-1)] * \underbrace{\delta(t+2)}_{\delta \text{ is even}} = \underbrace{u(t-3) * \delta(t+2) - u(t-1) * \delta(t+2)}_{\text{convolution is distributive}}$

0.5

$= u(t) * \delta(t-3) * \delta(t+2) - u(t) * \delta(t-1) * \delta(t+2)$

$= u(t) * \delta(t-1) - u(t) * \delta(t+1) = \cancel{\delta(t-1) * u(t) - \delta(t+1) * u(t)} = \cancel{\int_{-\infty}^{\tau} \delta(\tau-1)\,d\tau - \int_{-\infty}^{\tau} \delta(\tau+1)\,d\tau}$

$= \underbrace{u(1) * \delta(t-1) - u(-1) * \delta(t+1)}_{\text{sifting property}} = 1 * \delta(t-1) - 0 * \delta(t+1) = 1 * \delta(t-1) = \int_{-\infty}^{\tau} \delta(\tau-1)\,d\tau = \boxed{u(t-1)}$

2. a- $x(t) = \{ u(t-3)$ $y(t) = \begin{cases} x(t) & t \ge 1/2 \\ -x(-t) & \text{otherwise} \end{cases}$

$y(t) = \begin{cases} u(t-3) & t \ge 1/2 \\ -u(-t-3) & \text{otherwise} \end{cases}$ $u(t-3) = \begin{cases} 1 & 3 \le t \\ 0 & t < 3 \end{cases}$ $-u(-t-3) = \begin{cases} -1 & t \le -3 \\ 0 & -3 < t \end{cases}$

$\therefore y(t) = \begin{cases} 1 & 3 \le t \\ 0 & -3 < t < 3 \\ -1 & t \le -3 \end{cases}$

~~if $0 < t < \frac{1}{2}$, $y(t) = -u(-t-3)$~~

$\therefore$ for $3 \le t$, $y(t) = 1$ $y(-t) = -1 = -y(t)$

for $t \le -3$, $y(t) = -1$ $y(-t) = 1 = -y(t)$

for $-3 < t < 3$, $y(t) = 0$ $y(-t) = 0 = y(t)$

$\boxed{\text{so } y \text{ is neither even or odd } \forall t}$

if $y(t) = 0 = y(-t)$ then $y(t) = -y(-t)$

1.5

Problem 2 (5 points) The following questions are not related.

1. A LTI system has impulse response $h(t) = 3\frac{d}{dt}\delta(t) + 3\delta(t+1)$, where $\delta(t)$ is the delta function. Can you write input-output equations that describe this system?

2. A LTI system, when the input is an unknown $x(t)$, outputs the y(t) that is depicted in Fig. 1. Assume now that the input is $x_1(t)$ with corresponding output $y_1(t)$, and we know that $x_1(t) = 2x(t-1) + x(t+3)$. Calculate what is $y_1(5)$.

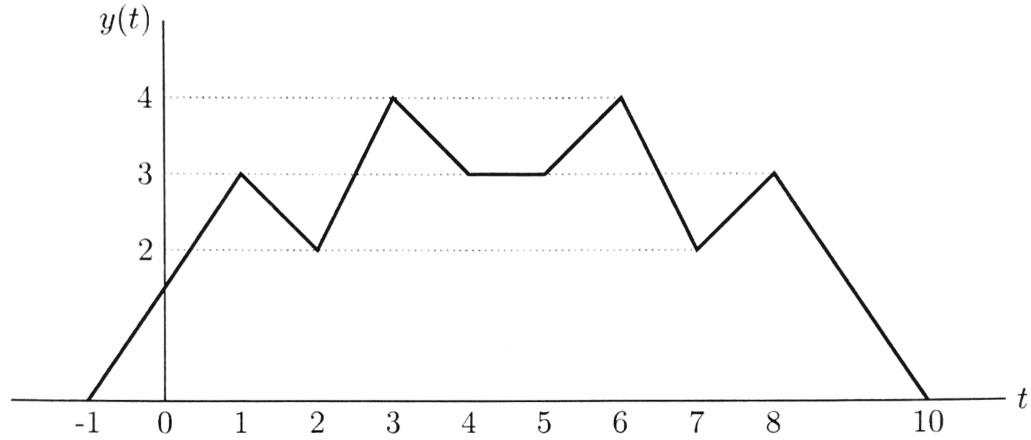
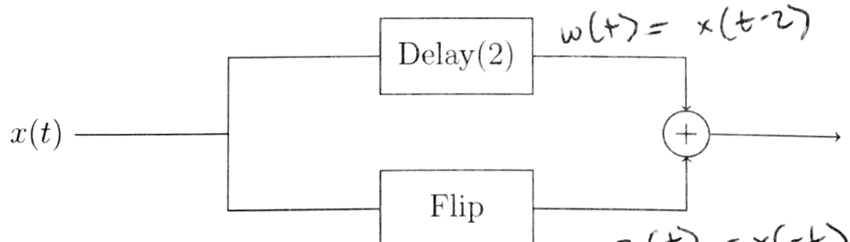


Figure 1: Response $y(t)$ for input $x(t)$

3. A system is described as depicted in Fig. 2, where is the Delay operator time shifts the input signal with the specified amount, and the Flip operator does the time reversal across Y-axis. Is this system causal? What about BIBO stability?



$w(t) = x(t-2)$

$y(t) = w(t) + z(t)$

$= x(t-2) + x(-t)$

$z(t) = x(-t)$

Figure 2: System for Problem 2(3).

(i) For an LTI system: $x(t) \to y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

$= \int_{-\infty}^{\infty} 3\frac{d}{d\tau}\delta(\tau)\, x(t-\tau)d\tau + \int_{-\infty}^{\infty} 3x(\tau)\delta(t-\tau+1) = \int_{-\infty}^{\infty} 3\frac{d}{dt}\delta(\tau)\, x(t-\tau)d\tau + 3\int_{-\infty}^{\infty} x(\tau)\delta(\tau-(t+1))d\tau$

$= 3\int_{-\infty}^{\infty} \frac{d}{dt}\delta(\tau)\, x(t-\tau)d\tau + 3x(t+1)\int_{-\infty}^{\infty}\delta(\tau-(t+1))d\tau = 3\int_{-\infty}^{\infty} \frac{d\delta(\tau)}{d\tau} x(t-\tau)d\tau + 3x(t+1)u(t+1)$

$\frac{d\delta(\tau)}{d\tau} = \delta(\tau)$ $\therefore$ $= 3x(t) + 3x(t+1)u(t+1)$

(2) ~~x(t) is LTI ? αx(t)~~

System is LTI $\therefore$ $Ax_a(t) + Bx_b(t) \rightarrow Ay_a(t) + By_b(t)$

AND $x(t-t_0) \rightarrow y(t-t_0)$

$\therefore$ if $x_1(t) = 2x(t-1) + x(t+3)$

$y_1(t) = 2y(t-1) + y(t+3)$

1.5

$\therefore y_1(5) = 2y(4) + y(8) = 2[3] + [3] = \boxed{9}$

(3) See figure : shows $y(t) = x(t-2) + x(-t)$ for $y(-2)$ it depends on $x(2)$.

$\therefore$ for any $t>0$, $y(t)$ doesn't depend on anything $>t$

i.e. on any time in the future ... $\therefore$ the system is ~~cont~~ causal

Note also delay & reflection are LTI $\therefore$ sum of systems is LTI. Reflection is not LTI

since $\delta(t-2) + \delta(-t) = 0$ for all $t<0$, system is causal

Also if $y(t) = x(t-2) + x(-t)$ $\therefore$ if $|x(t)| < A \;\forall\; t$,

1.5 then $|x(m) + x(n)| < B = 2A \;\forall$ ~~t~~ m & n

$\therefore$ $m = t-2$ & $n = -t$ $\therefore$

system is BIBO

Also since LTI,

$$\int_{-\infty}^{\infty} \left(\delta(t-2) + \delta(-t)\right) dt = \int_{-\infty}^{\infty} \delta(t-2)\,dt + \int_{-\infty}^{\infty} \delta(t)\,dt$$

$= 2 < \infty$ $\therefore$ BIBO

as system is not LTI

(1) $x(t) * h(t) = x(t) * 3\frac{d}{dt}\delta(t) + x(t) * \delta(t+1)$

$= x(t) * 3\frac{d}{dt}\delta(t) + 3x(t+1)$

$3 \cdot \frac{dx(t)}{dt}$

X $\frac{d}{dt}\delta(t) = \delta(t)$ $\therefore$ ~~$x(t)$~~

$$\boxed{x(t) \rightarrow y(t) = 3x(t) + 3x(t+1)}$$

1.5

Check solutions.

Problem 3 (5 points) Assume that $x_1(t)$ and $x_2(t)$ are two periodic signals, and both of them have period T. The convolution of $x_1(t)$ and $x_2(t)$ is not well defined, because we could be collecting infinite area with the integration. Instead, for periodic signals with period T, we use what is called the "periodic convolution", that is defined as follows:

$$y(t) = (x_1 \star x_2)(t) = \int_0^T x_1(\tau)x_2(t-\tau)d\tau$$

and where $\star$ stands for periodic convolution.

1. Assume both $x_1(t)$ and $x_2(t)$ are odd signals. Is $y(t) = (x_1 \star x_2)(t)$ even, odd, or neither? Justify your answer.

2. Let $x_2(t)$ be a periodic signal with period $T_1 = 5$, and $x_1(t)$ be the periodic sampling signal

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-4-5k),$$

where $\delta(t)$ denotes the delta function. Calculate the periodic convolution between $x_1(t)$ and $x_2(t)$ that is $(x_1 \star x_2)(t)$ as a function of $x_2(t)$.

1. $y(-t) = (x_1 \star x_2)(-t) = \int_0^T x_1(\tau)x_2(-t-\tau)d\tau$ ✓

let $u = t+\tau$ $du = d\tau$ $= \int_t^{t+T} x_1(u-t)\, x_2(u)\, du$

$= \int_t^{t+T} (-x_1(t-u))(-x_2(u))du = \int_t^{t+T} x_1(t-u)x_2(u)\,du$

$= \underbrace{\int_t^{t+T} x_1(u)x_2(t-u)du}_{\text{commutativity}} = \underbrace{\int_0^T x_1(u)x_2(t-u)\,du}_{\text{can be integrated over any period}} = y(t)$

$\boxed{\therefore\ y(t) \text{ is even}}$

✓

2. note, $x_1(t)$ has period $= T = 5$

$$\therefore \cancel{\text{the}} \quad (x_1 * x_2)(t) = \int_0^5 \left[\sum_{k=-\infty}^{\infty} \delta(\tau-4-5k)\right] x_2(t-\tau)\,d\tau$$

Since τ only varies from 0 to 5

$$\sum_{k=-\infty}^{\infty} \delta(\tau-4-5k) \underbrace{\approx}_{\text{is essentially equal to}} \delta(\tau-4)$$ since $x_1(\tau)=0$ for all values of $k \neq 0$ for all values of τ between 0 & 5

$$\therefore (x_1 * x_2)(t) = \int_0^5 \delta(\tau-4)\, x_2(t-\tau)\,d\tau = \int_0 x_2(t)\,\delta(t-\tau-4)\,d\tau$$

$$= \int_0^5 x_2(\tau)\,\delta(\tau-t+4)\,d\tau = \int_0 (x_2(\tau)\,\delta(\tau-(t-4))\,d\tau)$$

δ is even

$$= \int_0^5 x_2(t-4)\,\delta(\tau-(t-4))\,d\tau$$

$$= x_2(t-4)\int_0^5 \delta(\tau-(t-4))\,d\tau \quad \cancel{= x_2(t-4)\,u(}$$

$$\therefore (x_1 * x_2)(t) = \delta(t-4) * x_2(t) = \boxed{x_2(t-4)}$$