Total: 15 points

EE102: Signals and Systems

Midterm Exam

8:05 am - 9:35 am, October 23, 2017

NAME: ____________________ **UID**: ____________________

This exam has 3 problems, for a total of 15 points.

Closed book. No calculators. No electronic devices.
One page, letter-size, one-side cheat-sheet allowed.
Answer the questions in the space provided below each problem. If you run out of room for an answer, continue on the back of the page or use the extra pages at the end.
Please, write your name and UID on the top of each loose sheet!
GOOD LUCK!

Problem	Points	Total Points
1		5
2		5
3		5
Total		15

Extra Pages: __________

To fill in, in case extra sheets are used apart from what is provided.

Note: Answers without justification will not be awarded any marks.

Problem 1(5 points) The following questions are not related to each other.

1. In the following expressions, $*$ stands for the convolution operation, $u(t)$ is the unit step function and $\delta(t)$ is the delta function.

   (a) Simplify $\frac{du(t)}{dt} * (te^{-6|t+1|}\delta(t-3))$.

   (b) Calculate $\int_{-\infty}^{\infty} f(t)g(t)\,dt$ with $f(t) = u(t+3)$ and $g(t) = u(-t+5)$.

   (c) Calculate $y(0)$, where $y(t) = x_1(t) * x_2(t)$, $x_1(t) = u(t-3) - u(t-1)$ and $x_2(t) = \delta(-2-t)$.

2. Consider the signal

$$x(t) = u(t-3)$$

   and the signal

$$y(t) = \begin{cases} x(t) & \text{if} \quad t \geq 0.5 \\ -x(-t) & \text{otherwise} \end{cases}.$$

   Is the signal $y(t)$ even or odd or neither? Justify your answer.

Solution

1. (a)

$$\begin{aligned}\frac{du(t)}{dt} * (te^{-6|t+1|}\delta(t-3)) &= \delta(t) * (te^{-6|t+1|}\delta(t-3)) \\ &= te^{-6|t+1|}\delta(t-3) \\ &= 3e^{-24}\delta(t-3).\end{aligned}$$

   (b)

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g(t)\,dt &= \int_{-\infty}^{\infty} u(t+3)u(-t+5)\,dt \\ &= \int_{-3}^{5} u(t+3)u(-t+5)\,dt \\ &= \int_{-3}^{5} 1\,dt \\ &= 8.\end{aligned}$$

(c)

$$
\begin{aligned}
y(0) &= \int_{-\infty}^{\infty} x_1(\tau)x_2(-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} (u(\tau-3)-u(\tau-1))\,\delta(-2+\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} (u(-1)-u(1))\,\delta(-2+\tau)\,d\tau \\
&= -\int_{-\infty}^{\infty} \delta(-2+\tau)\,d\tau \\
&= -1.
\end{aligned}
$$

2. We can write the signal as

$$
\begin{aligned}
y(t) &= x(t)u(t-0.5) - x(-t)u(-t+0.5) \\
&= u(t-3)u(t-0.5) - u(-t-3)u(-t+0.5) \\
&= u(t-3) - u(-t-3).
\end{aligned}
$$

Now, we have that $y(-t) = u(-t-3) - u(t-3) = -y(t)$, thus it is odd.

Problem 2 (5 points) The following questions are not related.

1. A LTI system has impulse response $h(t) = 3\frac{d}{dt}\delta(t) + 3\delta(t+1)$, where $\delta(t)$ is the delta function. Can you write input-output equations that describe this system?

2. A LTI system, when the input is an unknown $x(t)$, outputs the y(t) that is depicted in Fig. 1. Assume now that the input is $x_1(t)$ with corresponding output $y_1(t)$, and we know that $x_1(t) = 2x(t-1) + x(t+3)$. Calculate what is $y_1(5)$.

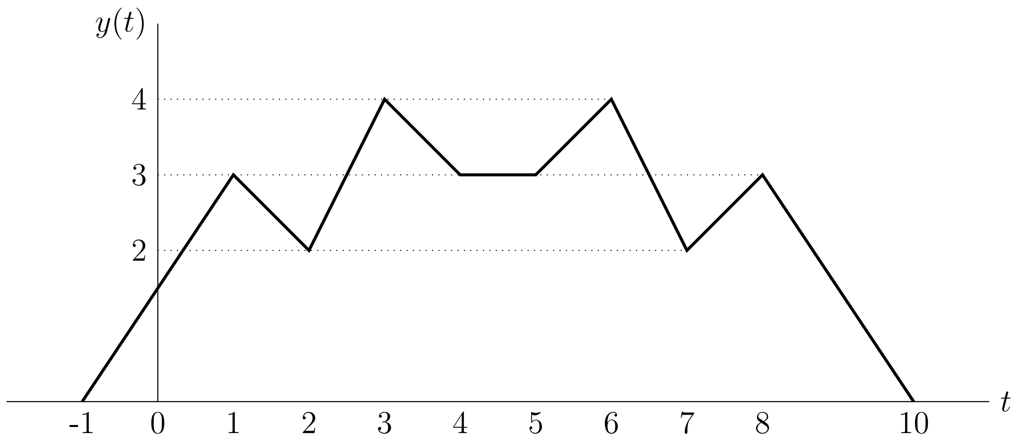


Figure 1: Response $y(t)$ for input $x(t)$

3. A system is described as depicted in Fig. 2, where is the Delay operator time shifts the input signal with the specified amount, and the Flip operator does the time reversal across Y-axis. Is this system causal? What about BIBO stability?

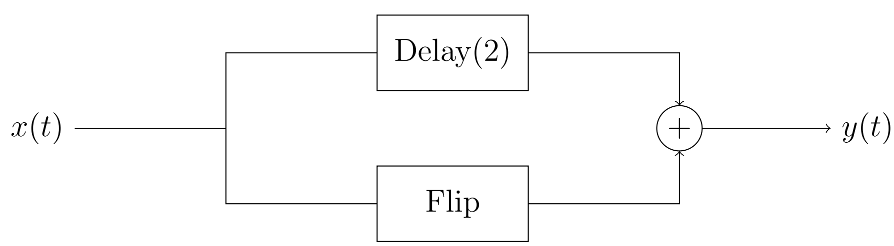


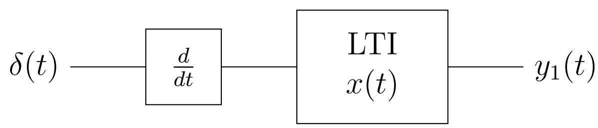
Figure 2: System for Problem 2(3).

Solution

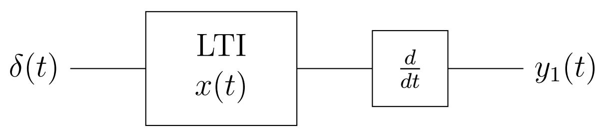
1.

$$
\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= x(t) \star \left(3\frac{d}{dt}\delta(t) + 3\delta(t+1) \right) \\
&= 3 \left(x(t) \star \frac{d}{dt}\delta(t) \right) + 3x(t+1) \\
&= 3y_1(t) + 3x(t+1).
\end{aligned}
$$

To compute $\left(x(t) \star \frac{d}{dt}\delta(t)\right)$, we write this in the following block diagram:



Since, both the system $\frac{d}{dt}$, and LTI system $x(t)$ are LTI systems; we can change their order.



Thus, $y_1(t) = \frac{d}{dt}\left(x(t) \star \delta(t)\right) = \frac{d}{dt}x(t)$. With this,

$$y(t) = 3y_1(t) + 3x(t+1) = 3\frac{d}{dt}x(t) + 3x(t+1).$$

2. As the system is linear and time invariant, the response of $x_1(t)$ will be

$$y_1(t) = 2y(t-1) + y(t+3).$$

Thus, $y_1(5) = 2y(4) + y(8) = 2 * 3 + 3 = 9$.

3.

$$y(t) = x(t-2) + x(-t).$$

The system is $\boxed{\text{not causal}}$ as $y(-2) = x(-4) + x(2)$ and thus it requires knowledge about the future input.

The system is $\boxed{\text{BIBO stable}}$ as

$$\begin{aligned}|y(t)| &= |x(t-2) + x(-t)| \\ &\leq |x(t-2)| + |x(-t)|.\end{aligned}$$

Thus if $|x(t)| \leq B$, for all t, for some finite B, then

$$|y(t)| \leq 2B.$$

Problem 3 (5 points) Assume that $x_1(t)$ and $x_2(t)$ are two periodic signals, and both of them have period T. The convolution of $x_1(t)$ and $x_2(t)$ is not well defined, because we could be collecting infinite area with the integration. Instead, for periodic signals with period T, we use what is called the "periodic convolution", that is defined as follows:

$$y(t) = (x_1 \star x_2)(t) = \int_0^T x_1(\tau)x_2(t-\tau)d\tau$$

and where $\star$ stands for periodic convolution.

1. Assume both $x_1(t)$ and $x_2(t)$ are odd signals. Is $y(t) = (x_1 \star x_2)(t)$ even, odd, or neither? Justify your answer.

2. Let $x_2(t)$ be a periodic signal with period $T_1 = 5$, and $x_1(t)$ be the periodic sampling signal

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-4-5k),$$

where $\delta(t)$ denotes the delta function. Calculate the periodic convolution between $x_1(t)$ and $x_2(t)$ that is $(x_1 \star x_2)(t)$ as a function of $x_2(t)$.

Solution

1. We are going to evaluate $y(-t)$, which give us

$$\begin{aligned} y(-t) &= \int_0^T x_1(\tau)x_2(-t-\tau)\ d\tau \\ &\overset{(a)}{=} -\int_0^T x_1(\tau)x_2(t+\tau)\ d\tau \\ &\overset{(b)}{=} \int_0^T x_1(-\tau)x_2(t+\tau)\ d\tau, \end{aligned}$$

where (a) and (b) follow from the fact that $x_2(t)$ and $x_1(t)$ are odd functions. We now do the change of variables $\tau' = -\tau$ (this gives us that $d\tau' = -d\tau'$). If we substitute this, in the above equation, we get that

$$y(-t) = -\int_0^{-T} x_1(\tau')x_2(t-\tau')\ d\tau' = \int_{-T}^0 x_1(\tau')x_2(t-\tau')\ d\tau',$$

Since $x_1(\tau')$ and $x_2(\tau')$ periodic with period T, then $x_1(\tau')x_2(t-\tau')$ is also periodic with period T for any t. Thus, we have that

$$\int_{-T}^0 x_1(\tau')x_2(t-\tau')\ d\tau' = \int_0^T x_1(\tau')x_2(t-\tau')\ d\tau' = y(t).$$

Thus, we have that $y(-t) = y(t)$, i.e., $y(t)$ is even.

2. Note that for $0 \leq \tau \leq 5$, the function $\delta(\tau - 4 - 5k) = 0$ for all integer $k \neq 0$. Therefore, for $0 \leq \tau \leq 5$, we have that $x_1(\tau) = \sum_{k=-\infty}^{\infty} \delta(\tau - 4 - 5k) = \delta(\tau - 4)$. With this in mind, let us evaluate the convolution $x_1(t) \star x_2(t)$ as follows for any t

$$
\begin{aligned}
x_1(t) \star x_2(t) &= \int_0^5 x_1(\tau) x_2(t-\tau) d\tau \\
&= \int_0^5 \delta(\tau - 4) x_2(t-\tau) d\tau \\
&= \int_0^5 \delta(\tau - 4) x_2(t-4) d\tau \\
&= x_2(t-4) \int_0^5 \delta(\tau - 4) d\tau \\
&= x_2(t-4) \int_{-4}^1 \delta(\tau') d\tau' = x_2(t-4) \ \times \ 1.
\end{aligned}
$$

Therfore, we have that $x_1(t) \star x_2(t) = x_2(t-4)$.