Total: 60 points

**Signals and Systems**

Final Exam
June 8, 2016

Problem 1 (12 points)

Consider a Linear Time Invariant (LTI) system with impulse response

$$h_1(t) = \delta(t) - 3\delta(t-4).$$

Assume we connect it in series with another system, that is also LTI and causal, and has the impulse response:

$$h_2(t) = e^{-t}\cos(3\pi t)u(t-\frac{1}{6}).$$

Find the transfer function $H(s)$ of the overall equivalent system. Deduce the frequency response of the overall equivalent system $H(\omega)$. Explain your answer.

**Solutions**: The overall impulse response is

$$h(t) = h_1(t) \star h_2(t) = h_2(t) - 3h_2(t-4)$$

Thus

$$H(s) = (1 - 3e^{-4s})H_2(s)$$

Let us now compute $H_2(s)$. We can equivalently write $h_2(t)$ as:

$$\begin{aligned} h_2(t) &= e^{-\frac{1}{6}}e^{-(t-\frac{1}{6})}\cos\left(3\pi(t-\frac{1}{6}) + \frac{\pi}{2}\right)u(t-\frac{1}{6}) \\ &= -e^{-\frac{1}{6}}e^{-(t-\frac{1}{6})}\sin\left(3\pi(t-\frac{1}{6})\right)u(t-\frac{1}{6}) \end{aligned}$$

Thus,

$$H_2(s) = -e^{-\frac{1}{6}}e^{-\frac{1}{6}s}\frac{3\pi}{(s+1)^2 + (3\pi)^2}$$

where $\text{Re}\{s\} > -1$. Thus

$$H(s) = -e^{-\frac{1}{6}}e^{-\frac{1}{6}s}(1 - 3e^{-4s})\frac{3\pi}{(s+1)^2 + (3\pi)^2}$$

where $\text{Re}\{s\} > -1$. Since the ROC includes the $j\omega$−axis, we can conclude that $H(\omega) = H(s)|_{s=j\omega}$, so that

$$H(\omega) = -e^{-\frac{1}{6}}e^{-\frac{1}{6}j\omega}(1 - 3e^{-4j\omega})\frac{3\pi}{(j\omega+1)^2 + (3\pi)^2}$$

Problem 2 (15 points) Problem on Fourier Series

1. Calculate the Fourier Series Coefficients for a periodic signal $x_1(t)$ with period $T = 5$ and values $x_1(t) = 1$, $0 \leq t < 1$, $x_1(t) = 2$, $1 \leq t < 2$, and $x_1(t) = 0$, $2 \leq t < 5$. $x_1(t)$ is shown in Figure 1.

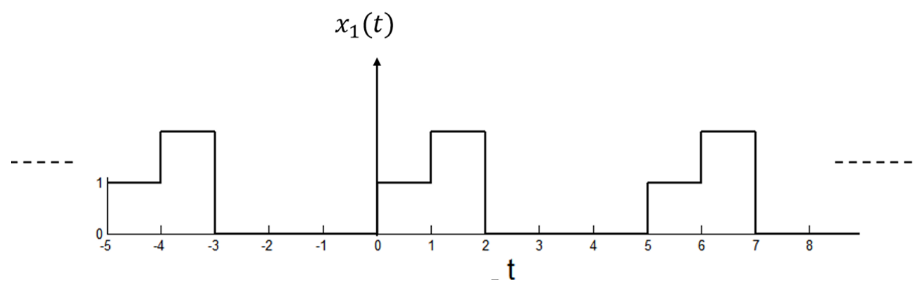



Figure 1: Signal $x_1(t)$

**Solutions**: $x_1(t)$ is periodic with period $T = 5$ and frequency $\Omega = 2\pi\frac{1}{T} = \frac{2\pi}{5}$. The Fourier Series coefficients of $x_1(t)$ are calculated as follows:

$$\begin{aligned} X_k^1 &= \frac{1}{T}\int_0^T x_1(t)e^{-j\Omega kt}dt \\ &= \frac{1}{5}\left(\int_0^1 e^{-j\Omega kt}dt + 2\int_1^2 e^{-j\Omega kt}dt\right) \\ &= \frac{1}{5}\left(\frac{e^{-j\Omega k} - 1}{-j\Omega k} + 2\frac{e^{-j2\Omega k} - e^{-j\Omega k}}{-j\Omega k}\right) \\ &= \frac{1}{5}\left(\frac{2e^{-j2\Omega k} - e^{-j\Omega k} - 1}{-j\Omega k}\right) \end{aligned}$$

2. Calculate the Fourier Series Coefficients for the signal $x_2(t) = x_1(t) + x_1(-t)$.

**Solutions**: The Fourier Series coefficients of $x_2(t)$ are calculated as follows:

$$\begin{aligned}
X_k^2 &= X_k^1 + X_{-k}^1 \\
&= \frac{1}{5}\left(\frac{2e^{-j2\Omega k} - e^{-j\Omega k} - 1}{-j\Omega k} + \frac{2e^{j2\Omega k} - e^{j\Omega k} - 1}{j\Omega k}\right) \\
&= \frac{1}{5}\left(\frac{4\sin(2\Omega k) - 2\sin(\Omega k)}{\Omega k}\right)
\end{aligned}$$

3. Calculate the Fourier Transform of $x_3(t) = x_1(t) \cdot x_2(t)$ (multiplication of the two signals).

   **Solutions**: $x_3(t)$ can be equivalently written as:

$$x_3(t) = x_1(t)x_1(t) + x_1(t)x_1(-t) = x_1(t)^2$$

$x_1(t)^2$ is a periodic signal of period 5 that has the same form of signal $x_1(t)$ but for $1 < t < 2$, it takes the value of $2^2 = 4$. Therefore the Fourier Series coefficients as given as follows:

$$\begin{aligned}
X_k^3 &= \frac{1}{T}\int_0^T x_3(t)e^{-j\Omega kt}dt \\
&= \frac{1}{5}\left(\int_0^1 e^{-j\Omega kt}dt + 4\int_1^2 e^{-j\Omega kt}dt\right) \\
&= \frac{1}{5}\left(\frac{e^{-j\Omega k} - 1}{-j\Omega k} + 4\frac{e^{-j2\Omega k} - e^{-j\Omega k}}{-j\Omega k}\right) \\
&= \frac{1}{5}\left(\frac{4e^{-j2\Omega k} - 3e^{-j\Omega k} - 1}{-j\Omega k}\right)
\end{aligned}$$

Problem 3 (8 points)

Assume you are given the Laplace transform $X(s)$ of the signal $x(t)$ depicted in Figure 1, and you are told that the Region of Convergence of $X(s)$ includes the y-axis ($i\omega$). Assume now that this signal is made periodic with period $T$ to create the signal $x_T(t)$, namely, $x(t)$ repeated with a period $T$ as depicted in Figure 2. Could you derive the Fourier Series coefficients for $x_T(t)$ as a function of $X(s)$ and $T$?

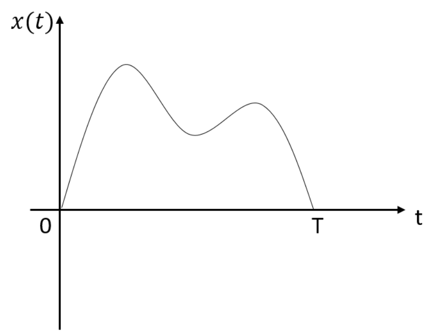



Figure 2: Signal $x(t)$

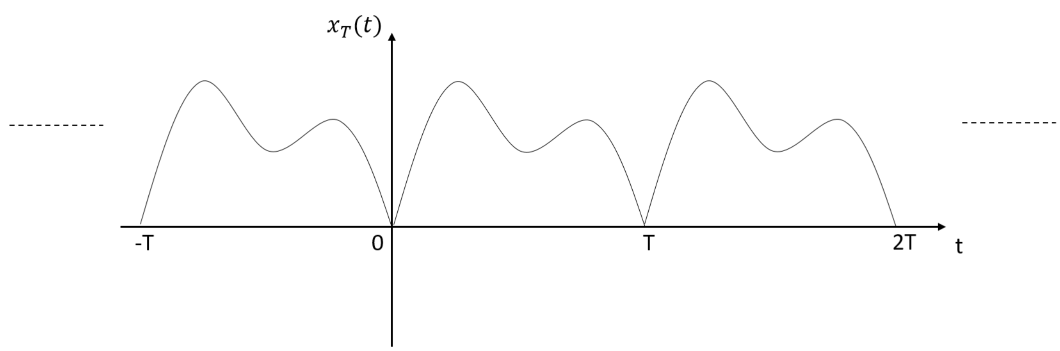



Figure 3: Signal $x_T(t)$

**Solutions**: The Laplace transform of $x(t)$ is given as follows:

$$X(s) = \int_0^T x(t)e^{-st}dt$$

On the other hands, the Fourier series coefficients of $x_T(t)$ are computed as follows:

$$C_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt$$

Since the ROC of $X(s)$ includes the $j\omega-$axis, we conclude that:

$$C_k = \frac{1}{T} X(s)|_{s=j\frac{2\pi k}{T}}$$

Problem 4 (9 points)

In this problem, you have three signals $x_1(t)$, $x_2(t)$ and $x_3(t)$. Their corresponding two-sided Laplace transforms $X_1(s)$, $X_2(s)$ and $X_3(s)$ have the following characteristics:

- $X_1(s)$ has two poles: $-1$ and 1 and no zeros;
- $X_2(s)$ has one pole at $-1$ and one zero at 2;
- $X_3(s)$ has two zeros: $-1$ and 1 and no poles.

1. If $x_1(t)$, $x_2(t)$ and $x_3(t)$ are impulse responses of three LTI systems. What conditions should the ROC of each Laplace transform satisfy so that each system is stable?
2. What conditions should the ROC of each Laplace transform satisfy so that each signal is 0 for $t > 0$?
3. Can $x_1(t)$ be an even signal? If so, what condition the ROC of $x_1(t)$ should satisfy? (First prove that for even signal $f(t)$, we should have $F(s) = F(-s)$ where $F(s)$ is the two-sided Laplace transform of $f(t)$ and where for any $s \in$ ROC of $F(s)$ we also have $-s \in$ ROC of $F(s)$).

**Solutions**:

1. For the system to be stable, the ROC should include the $j\omega-$axis.

   For $x_1(t)$, ROC1 should be: $-1 <\text{Re}\{s\} < 1$.

   For $x_2(t)$, ROC2 should be: $\text{Re}\{s\} > -1$.

   $x_3(t)$ has no poles, so the ROC is the whole s-plane.

2. For each signal to be zero for $t > 0$, the ROC should have the form: $\text{Re}\{s\} < a$.

   For $x_1(t)$, ROC1 should be: $\text{Re}\{s\} < -1$.

   For $x_2(t)$, ROC2 should be: $\text{Re}\{s\} < -1$.

   $x_3(t)$ has no poles, so the ROC is always the whole s-plane and the signal $x_3(t)$ already satisfies the condition required for this problem.

3. The Laplace transform of $f(t)$ is:

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

If $f(t)$ is even, then

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{\infty} f(-t)e^{-st}dt = \int_{-\infty}^{\infty} f(t)e^{st}dt = F(-s)$$

$X_1(s)$ has this form:

$$X_1(s) = \frac{1}{(s+1)(s-1)}$$

It can be easily seen that $X_1(s) = X_1(-s)$. But we also need to have $-s$ and $s$ in the ROC of $X_1(s)$. This is satisfied when $-1 <\text{Re}\{s\} < 1$.

Problem 5 (16 points)

(a) Consider the following system $\mathcal{S}$: $y(t) = \cos(\omega_c t)x(t)$, where $\omega_c \neq 0$.

Is this system LTI?

**Solutions**: the system is linear but not TI.

(b) In this problem, we are going to design a system that is equivalent to $\mathcal{S}$. This new system is represented as follows:

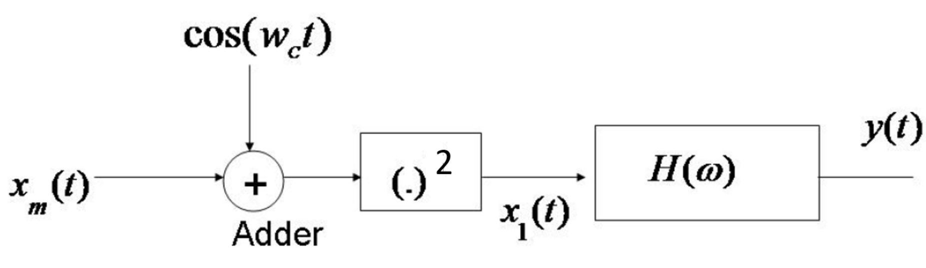



Figure 4: System in problem 5

where the block $(.)^2$ computes the square of its input (please watch out to the fact that $\cos(\omega_c t)$ is added to $x_m(t)$ not multiplied). Any input to this system $x_m(t)$ is

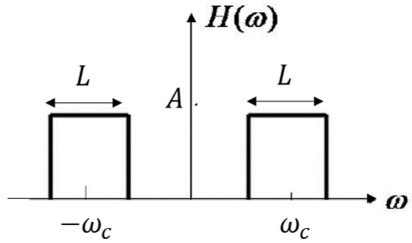



Figure 5: The bandpass filter of problem 5

assumed to be bandlimited, i.e., $X_m(\omega)$ is non-zero for $-\omega_M \leq \omega \leq \omega_M$. $H(\omega)$ is an ideal bandpass filter with amplitude $A$ as shown in Figure 5.

$L$ is the width of each rectangle and $\omega_c$ is the center of each rectangle.

Assume now that we apply to this system a specific $x_m(t)$ whose Fourier transform $X_m(\omega)$ is shown in the next page:

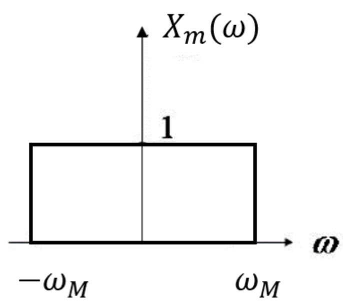



Figure 6: The input signal in Problem 5

1. Sketch $X_1(\omega)$ the Fourier transform of $x_1(t)$ ($x_1(t)$ is an intermediate signal shown in Figure 4).
2. How $\omega_c$ should be chosen in terms of $\omega_M$ so that no aliasing happens in the frequency domain. (In $X_1(\omega)$, the part coming from $X_m(\omega)$ is not distorted).
3. How $A$ and $L$ of the band-pass filter should be chosen so that $y(t) = \cos(\omega_c t)x_m(t)$?

**Solutions**:

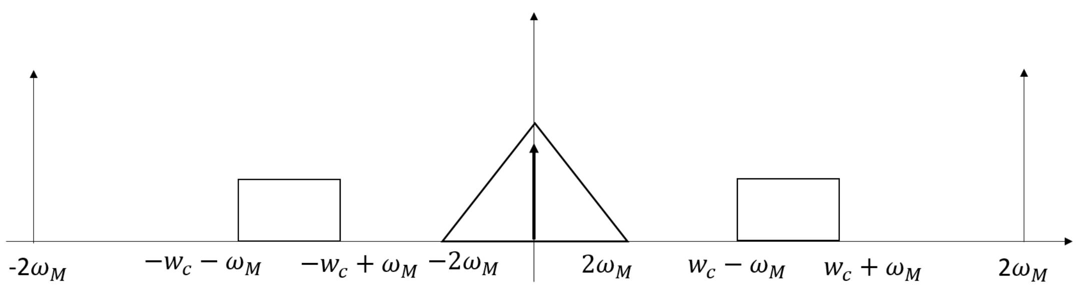
1. We have

$$\begin{aligned} x_1(t) &= (x_m(t) + \cos(\omega_c t))^2 \\ &= (x_m(t))^2 + 2x_m(t)\cos(\omega_c t) + \cos^2(\omega_c t) \\ &= (x_m(t))^2 + 2x_m(t)\cos(\omega_c t) + \frac{1+\cos(2\omega_c t)}{2} \end{aligned}$$

Therefore,

$$X_m(\omega) = \frac{1}{2\pi}X_m(\omega) \star X_m(\omega) + X_m(\omega - \omega_c) + X_m(\omega + \omega_c) + \pi\delta(\omega) + 2\pi\delta(\omega - 2\omega_c) + 2\pi\delta(\omega + 2\omega_c)$$

The following figure is sketch of $X_m(\omega)$:




2. We should have $\omega_c - \omega_M > 2\omega_M$ or $\omega_c > 3\omega_M$.

   We also should have $2\omega_c > \omega_c + \omega_M$ or $\omega_c > \omega_M$.

   Therefore we should have $\omega_c > 3\omega_M$.

3. $y(t) = \cos(\omega_c t)x_m(t)$ has the following Fourier transform: $Y(\omega) = \frac{1}{2}\left(X_m(\omega - \omega_c) + X_m(\omega + \omega_c)\right)$.

   Therefore, $A = \frac{1}{2}$

   and

   $2\omega_M < \omega_c - \frac{L}{2} < \omega_c - \omega_M$ or $2\omega_M - \omega_c < -\frac{L}{2} < -\omega_M$ thus

   $2\omega_M < L < 2(\omega_c - 2\omega_M)$