Table 1: Score Table

Problem	a	b	c	d	e	Score	
1	5	5				10	10
2	2	2	2	5	2	13	9
3	5	2	5			12	12
4	10					10	10
5	5	5	5			15	13
6	7	8				15	15
Total						75	69

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{Re}[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{Re}[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{Re}[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
9.	$2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{Re}[s] > -a$
10.	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{Re}[s] > 0$
11.	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{Re}[s] > -a$
12.	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{Re}[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\, f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

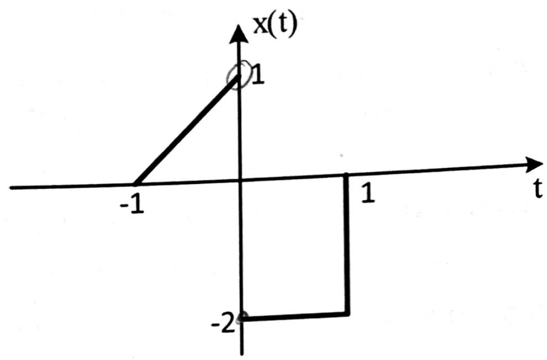
Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Problem 1 (10 pts)

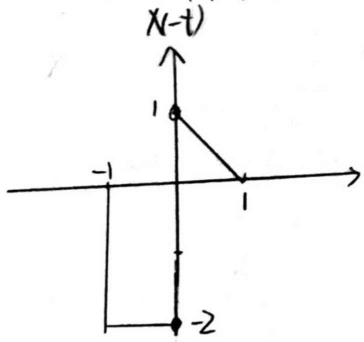
Consider signal $x(t)$ depicted in the figure below



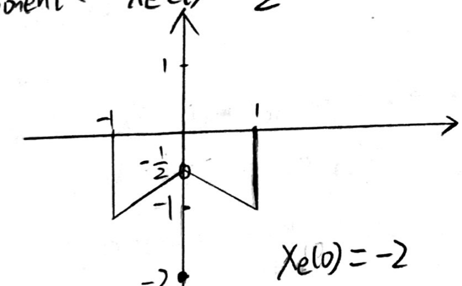
(a) (5 pts) Sketch even and odd components of $x(t)$. Assume $x(0) = -2$.

(b) (5 pts) Sketch $2x(-2t-1)$.

a)



even Component: $X_e(t) = \frac{1}{2}(x(t) + X(-t))$

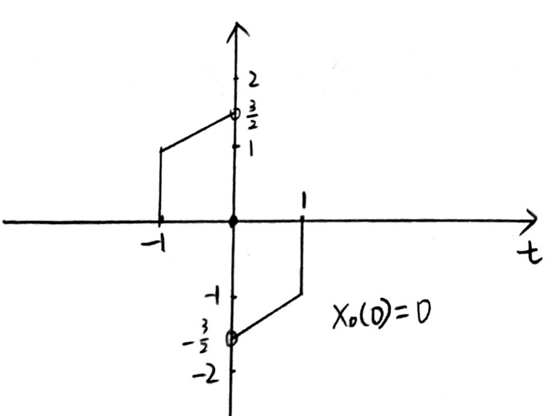


$X_e(0) = -2$

5

odd Component:

$X_o(t) = \frac{1}{2}(X(t) - X(-t))$



$X_o(0) = 0$

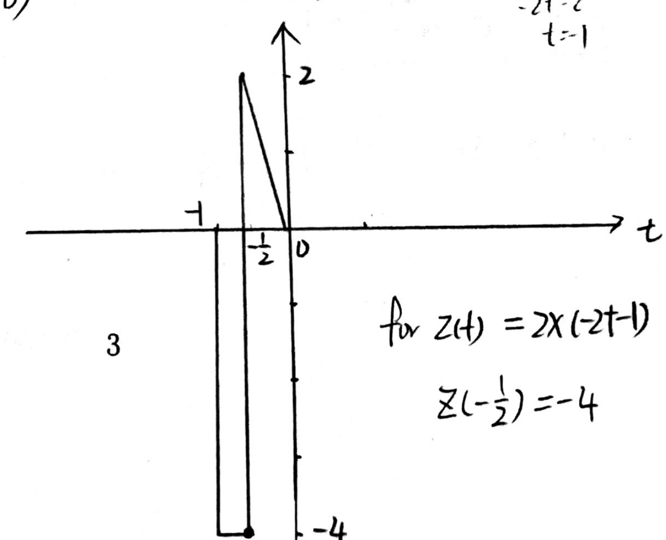
b)

$2X(-2t-1)$

$-2t-1=0$

$-2t=2$

$t=-1$



for $Z(t) = 2X(-2t-1)$

$Z(-\frac{1}{2}) = -4$

5

9

Problem 2 (13 pts)

The system S is given by the following relation

$$y(t) = x(t-2) + x(2-t), \quad -\infty < t < \infty$$

1) (6 pts) Using the system relation, answer the following questions about the system. You need to justify your answer.

   (a) (2 pts) Prove that the system is linear.

   (b) (2 pts) Is the system time-varying or time-invariant?

   (c) (2 pts) Is the system causal or not causal?

2) (5 pts) Find the impulse response function of the system.

3) (2 pts) Verify your answers for parts b) and c) in 1) using the impulse response function of the system. You need to justify your answer.

1) a) let $a(t), b(t) \in S$, and α, β are constants.

$$a(t) \rightarrow \boxed{S} \rightarrow Y_a(t) = a(t-2) + a(2-t)$$

$$b(t) \rightarrow \boxed{S} \rightarrow Y_b(t) = b(t-2) + b(2-t)$$

Then $\alpha a(t) + \beta b(t) \rightarrow \boxed{S} \rightarrow S[\alpha a(t) + \beta b(t)] = \alpha a(t-2) + \beta b(t-2) + \alpha a(2-t) + \beta b(2-t)$

$$= \alpha[a(t-2) + a(2-t)] + \beta[b(t-2) + b(2-t)]$$

2

$$= \alpha S[a(t)] + \beta S[b(t)] \Rightarrow \boxed{\text{System is linear.}}$$

b) $x(t) \rightarrow \boxed{S} \rightarrow Y(t) = x(t-2) + x(2-t)$

$$x(t-\tau) \rightarrow \boxed{S} \rightarrow S[x(t-\tau)] = x(t-\tau-2) + x(2-t+\tau) = Y(t-\tau)$$

1

$\boxed{\text{The system is TI}}$

c) The system is $\boxed{\text{not causal}}$ because for $t<0$, $Y(t) = x(t-2) + x(2-t)$

x(2-t) depend on 2-t which is greater than t..

2

So the system is not causual 5

2) $h(t) = \delta(t-2) + \delta(2-t)$, $-\infty < t < +\infty$

$h(t,\tau) = \delta(t-\tau-2) + \delta(2-t+\tau)$, $-\infty < t < +\infty$

3

3)

b) $h(t-\tau) = \delta(t-\tau-2) + \delta(2-t+\tau) = h(t,\tau)$

So the system is $\boxed{TI}$ since $h(t-\tau) = h(t,\tau)$

1

c) $h(t) \neq 0$ when $t<0$ due to $\delta(t-2)$

So the system is $\boxed{\text{not causal}}$.

Problem 3 (12 pts)

Consider an LTI system with the following input-output relationship:

$$y(t) = \int_{-\infty}^{t-1} e^t cos(2\tau + 2 - 2t)x(\tau)e^{-\tau+2}d\tau$$

where $x(t)$ and $y(t)$ are the input and the output of the system, respectively.

(a) (5 pts) Find the impulse response function $h(t)$.

(b) (2 pts) Is the system C or NC? Provide justification.

(c) (5 pts) Is this system BIBO stable? Provide justification.

a) $y(t) = \int_{-\infty}^{t-1} e^t \cos(2\tau+2-2t)\, x(\tau)\, e^{-\tau+2}\, d\tau$

$= \int_{-\infty}^{+\infty} e^{t-\tau+2} \cos(-2(t-\tau)+2)\, u(t-1-\tau)\, x(\tau)\, d\tau.$

By defn, $h(t,\tau) = e^{t-\tau+2} \cos(-2(t-\tau)+2)\, u(t-\tau-1)$

$\boxed{h(t) = e^{t+2} \cos(-2t+2)\, u(t-1)}$ √5

b) The system is $\boxed{\text{Causual}}$ because $h(t)=0$ for $t<0$ due to the $u(t-1)$ √2

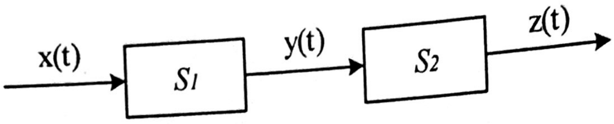
c) $\int_{-\infty}^{+\infty} |e^{t+2} \cos(-2t+2)\, u(t-1)|\, dt$

$= \int_{1}^{+\infty} e^{t+2} |\cos(-2t+2)|\, dt < \int_{1}^{+\infty} e^{t+2}\, dt = [e^{t+2}]_1^{+\infty} = +\infty$

The system is not BIBO stable √5

Problem 4 (10 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.



The first system S_1 is described by:

$$y(t) = \int_0^t e^{-\sigma} t x(\sigma) d\sigma, \quad t > 0$$

where $x(t)$ and $y(t)$ are the input and the output, respectively. The second system is described by:

$$z(t) = \int_2^t \sigma y(\sigma) d\sigma, \quad t > 2$$

where $y(t)$ and $z(t)$ and the input and the output, respectively.

Find the impulse response function $h_{12}(t, \tau)$ of the cascaded system S_{12}.

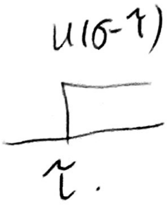
$$h_1(t,\tau) = \int_0^t e^{-\sigma} t\, \delta(\sigma-\tau)\, d\sigma = t \int_0^t e^{-\sigma} \delta(\sigma-\tau)\, d\sigma = t \int_0^t e^{-\tau} \delta(\sigma-\tau)\, d\sigma$$

$$= t e^{-\tau} \int_0^t \delta(\sigma-\tau)\, d\sigma = t e^{-\tau} u(\tau)\, u(t-\tau) \quad \checkmark_4$$

$$h_{12}(t,\tau) = \int_2^t \sigma\, h_1(\sigma,\tau)\, d\sigma = \int_2^t \sigma(\sigma e^{-\tau} u(\tau) u(\sigma-\tau))\, d\sigma$$

$$= e^{-\tau} \int_2^t \sigma^2 u(\tau)\, u(\sigma-\tau)\, d\sigma = e^{-\tau} u(\tau) \int_2^t \sigma^2 u(\sigma-\tau)\, d\sigma$$

$$= \begin{cases} e^{-\tau} u(\tau) \int_2^t \sigma^2 d\sigma & \text{for } \tau < 2 \\ e^{-\tau} u(\tau) \int_\tau^t \sigma^2 d\sigma & \text{for } 2 \le \tau \le t \\ 0 & \text{for } \tau > t \end{cases}$$



$$
= \begin{cases} e^{-\tau} u(\tau) \left[\frac{1}{3}\sigma^3\right]_2^t & \text{for } \tau < 2 \\ e^{-\tau} u(\tau) \left[\frac{1}{3}\sigma^3\right]_\tau^t & \text{for } 2 \le \tau \le t \\ 0 & \text{for } \tau > t \end{cases}
$$

$$
= \begin{cases} \frac{1}{3} e^{-\tau} (t^3 - 8)\, u(\tau) & \text{for } \tau < 2 \\ \frac{1}{3} e^{-\tau} (t^3 - \tau^3)\, u(\tau) & \text{for } 2 \le \tau \le t \\ 0 & \text{for } \tau > t \end{cases}
$$

$\checkmark$ 6

$$
= \begin{cases} \frac{1}{3} e^{-\tau} (t^3 - 8)\, u(\tau)\, u(t-\tau) & \text{for } \tau < 2 \\ \frac{1}{3} e^{-\tau} (t^3 - \tau^3)\, u(\tau)\, u(t-\tau) & \text{for } 2 \le \tau \end{cases}
$$

10

Problem 5 (15 pts)

1) (5 pts) Find the Laplace transform and ROC of the following function

$f(t) = te^{-3t}(sin(t) + cos(t))^2 u(t).$

Hint: $sin(2x) = 2sin(x)cos(x).$

2) (10 pts) Find the functions that correspond to the following Laplace transforms. Assume that time-domain functions are causal.

(a) (5 pts) $F(s) = \frac{s}{(s+1)^2+3}$

(b) (5 pts) $F(s) = \frac{1}{s^4}(1 - e^{-s})^2.$

1) $f(t) = te^{-3t}[\sin^2 t + \cos^2 t + 2\sin t \cos t]\, u(t)$

$= te^{-3t}[1 + \sin 2t]\, u(t)$

$F(s) = \mathcal{L}[te^{-3t}(1+\sin 2t)u(t)](s)$

$= -\frac{d}{ds}\mathcal{L}[e^{-3t}(1+\sin 2t)u(t)](s)$

$= -\frac{d}{ds}[\mathcal{L}[e^{-3t}u(t)](s) + \mathcal{L}[e^{-3t}\sin 2t\, u(t)](s)]$

$= -\frac{d}{ds}[\frac{1}{s+3} + \frac{2}{(s+3)^2+4}]$

$= -[-\frac{1}{(s+3)^2} - \frac{2(2s+6)}{(s^2+6s+13)^2}]$

5

$= \frac{1}{(s+3)^2} + \frac{4(s+3)}{(s^2+6s+13)^2}$

ROC: $Re\{s\} > -3$

2) a) $F(s) = \frac{s}{(s+1)^2+3} = \frac{s+1}{(s+1)^2+3} - \frac{1}{\sqrt{3}} \frac{\sqrt{3}}{(s+1)^2+3}$

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+3}\right](t) - \frac{1}{\sqrt{3}} \mathcal{L}^{-1}\left[\frac{\sqrt{3}}{(s+1)^2+3}\right](t)$$

$$= e^{-t}\cos(\sqrt{3}t) - \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t)$$

$$= \left[e^{-t}\cos(\sqrt{3}t) - \frac{\sqrt{3}}{3} e^{-t}\sin(\sqrt{3}t)\right] u(t)$$

5

b) $F(s) = \frac{1}{s^4}(1-e^{-s})^2 = \frac{1}{s^4}(1+e^{-2s}-2e^{-s})$

$$= \frac{1}{s^4} + \frac{e^{-2s}}{s^4} - \frac{2e^{-s}}{s^4}$$

$$= \frac{3!}{3!}\frac{1}{s^4} + \frac{3!}{3!}\frac{e^{-2s}}{s^4} - \frac{3!}{3!}\frac{2e^{-s}}{s^4}$$

$$= \frac{1}{6}\frac{3!}{s^4} + \frac{1}{6}e^{-2s}\frac{3!}{s^4} - \frac{1}{3}e^{-s}\frac{3!}{s^4}$$

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \left[\frac{1}{6}t^3 + \frac{1}{6}(t-2)^3 - \frac{1}{3}(t-1)^3\right] u(t)$$

$u(t-2)$ $u(t-1)$

3

Problem 6 (15 pts)

Consider an LTI system S with the following impulse response function

$$h(t) = (\frac{2}{3}e^{-t+1} + \frac{1}{3}e^{2t-2})u(t-1)$$

(a) (7 pts) Find the transfer function $H(s)$. Sketch its zero-pole plot and then denote ROC in the plot.

(b) (8 pts) If the output of the system is

$$y(t) = (-\frac{1}{5}cos(t-2) - \frac{2}{5}sin(t-2) + \frac{1}{5}e^{2(t-2)})u(t-2)$$

find its corresponding causal input $x(t)$.

a) $h(t) = (\frac{2}{3}e^{-t+1} + \frac{1}{3}e^{2t-2})\,u(t-1)$

$$H(s) = \int_{0^-}^{+\infty} (\frac{2}{3}e^{-t+1} + \frac{1}{3}e^{2t-2})\,u(t-1)\,e^{-st}\,dt$$

$$= \int_1^{+\infty} \frac{2}{3}e^{-(s+1)t+1} + \frac{1}{3}e^{-(s-2)t-2}\,dt$$

$$= \left[\frac{2}{3}(-\frac{1}{s+1})e^{-(s+1)t+1} + \frac{1}{3}(-\frac{1}{s-2})e^{-(s-2)t-2}\right]_1^{+\infty}$$

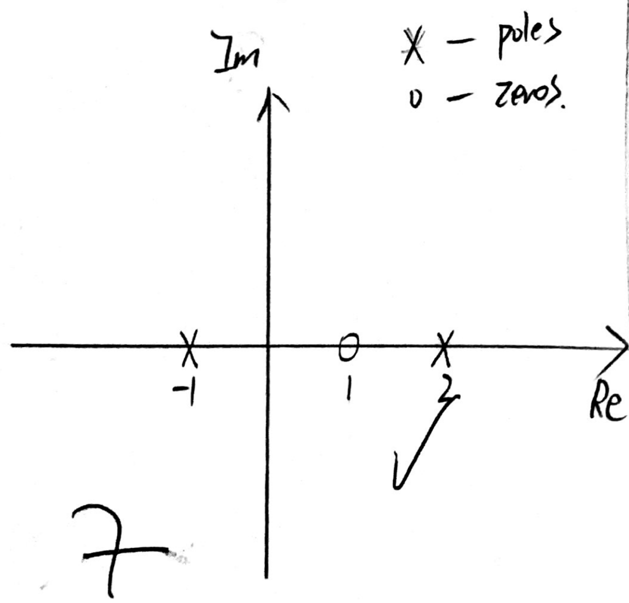
$$= \frac{2}{3}\frac{1}{s+1}e^{-s} + \frac{1}{3}\frac{1}{s-2}e^{-s}$$

$$= \frac{1}{3}e^{-s}(\frac{2}{s+1} + \frac{1}{s-2})$$

$$= \frac{1}{3}e^{-s}\frac{3(s-1)}{(s+1)(s-2)}$$

$$= e^{-s}\frac{s-1}{(s+1)(s-2)}$$ ROC: $Re\{s\} > 2$ ✓

Zero-pole plot.



7

b) $y(t) = \left(-\frac{1}{5}\cos(t-2) - \frac{2}{5}\sin(t-2) + \frac{1}{5}e^{2(t-2)}\right) u(t-2)$

$$Y(s) = \mathcal{L}[y(t)](s)$$

$$= -\frac{1}{5}e^{-2s}\frac{s}{s^2+1} - \frac{2}{5}e^{-2s}\frac{1}{s^2+1} + \frac{1}{5}e^{-2s}\frac{1}{s-2}$$

$$= \frac{1}{5}e^{-2s}\left[-\frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s-2}\right]$$

$$= \frac{1}{5}e^{-2s}\frac{5}{(s^2+1)(s-2)} = e^{-2s}\frac{1}{(s^2+1)(s-2)} \checkmark_1$$

Since for LTI system

$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s)$$

$$X(s) = \frac{Y(s)}{H(s)} \checkmark_1 = \frac{\frac{e^{-2s}}{(s^2+1)(s-2)}}{\frac{e^{-s}(s-1)}{(s+1)(s-2)}} = e^{-s}\frac{s+1}{(s^2+1)(s-1)} \checkmark_4$$

Note: $C = \left.\frac{s+1}{s^2+1}\right|_{s=1} = 1$

$$\frac{s+1}{(s^2+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1}$$

$$s+1 = As^2 - As + Bs - B + Cs^2 + C$$

$$s+1 = (A+C)s^2 + (B-A)s + (C-B)$$

$$\begin{cases} A+C=0 \\ B-A=1 \\ C-B=1 \end{cases} \quad \begin{cases} A=-1 \\ B=0 \\ C=1 \end{cases}$$

$$= e^{-s}\left[\frac{-s}{s^2+1} + \frac{1}{s-1}\right]$$

$$\boxed{x(t) = \mathcal{L}^{-1}[X(s)] = \left[-\cos(t-1) + e^{t-1}\right] u(t-1)} \checkmark_2$$

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