UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Practice Midterm Examination II- Solution

Write Your Discussion Session in the Corner→→↗↗
(* Otherwise Your Midterm might be LOST)

Your name:——————————————————

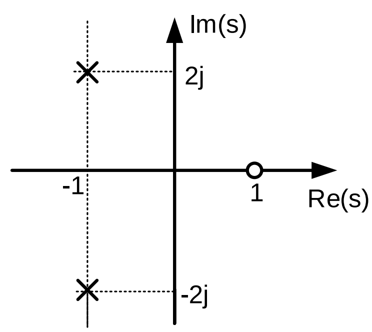
Instructions: Closed Book except one double-sided cheat sheet, Calculators are NOT Allowed

Good Luck!

Question 1

The figure below is the "poles-zeros" plot of a system transfer function $H(s)$.

(i) Find $H(s)$ given $H(0) = 1$.
(ii) Find Laplace transform of the output signal $Y(s)$ if input is $x(t) = e^{-2t}u(t)$.
(iii) Find the time domain output signal $y(t)$ from $Y(s)$.



Solution:

(i) The transform function is

$$H(s) = A\frac{(s-1)}{(s+1+2j)(s+1-2j)} = A\frac{s-1}{(s+1)^2+4} = A\frac{s-1}{s^2+2s+5}$$

Since $H(0) = -A/5 = 1$, therefore,$A = -5$.

$$H(s) = -5\frac{s-1}{s^2+2s+5}$$

(ii)

$$X(s) = \frac{1}{s+2}$$

thus

$$Y(s) = \frac{-5(s-1)}{(s+2)(s^2+2s+5)} = -5\left[\frac{B}{s+2} + \frac{Cs+D}{s^2+2s+5}\right]$$

Multiply both sides by $(s+2)(s^2+2s+5)$ and compare coefficients of powers of s on both sides to get $B = -3/5$, $C = 3/5$, and $D = 1$.

$$
\begin{aligned}
Y(s) =& A\left[\frac{B}{s+2} + \frac{C(s+1)+(D-C)}{(s+1)^2+(2)^2}\right] \\
Y(s) =& A\left[\frac{B}{s+2} + \frac{C(s+1)}{(s+1)^2+(2)^2} + \frac{D-C}{2}\frac{2}{(s+1)^2+(2)^2}\right]
\end{aligned}
$$

Take inverse Laplace transform to obtain

$$
y(t) = A\left[Be^{-2t} + Ce^{-t}\cos(2t) + \frac{D-C}{2}e^{-t}\sin(2t)\right]u(t)
$$

Substitute values of A, B, C and D to get

$$
\begin{aligned}
y(t) &= -5\left[-\frac{3}{5}e^{-2t} + \frac{3}{5}e^{-t}\cos(2t) + \frac{1}{5}e^{-t}\sin(2t)\right]u(t) \\
y(t) &= \left[3e^{-2t} - 3e^{-t}\cos(2t) - e^{-t}\sin(2t)\right]u(t)
\end{aligned}
$$

Question 2

Consider the following differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 25y(t) = \frac{dx(t)}{dt} + 3x(t), t > 0$$

with initial condition

$$y'(0) = 0, y(0) = 0, x(0) = 0.$$

(i) Compute $H(s)$ and $h(t)$.
(ii) Find poles and zeros of $H(s)$.
(iii) Is the system BIBO stable? Why?
(iv) Find input signal if the output signal is $y(t) = e^{-3t}u(t)$.
Solution:

(i) Taking Laplace Transform to both sides

$$s^2Y(s) + 6sY(s) + 25Y(s) = sX(s) + 3X(s).$$

The transform function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2+6s+25} = \frac{s+3}{(s+3)^2+4^2}.$$

$$h(t) = \cos(4t)e^{-3t}u(t).$$

(ii) One zero at $s = -3$ and a pair of complex poles at $s = -3 \pm 4j$.
(iii) System is BIBO stable since all poles are in the open left-s plane.
(iv) Since

$$Y(s) = \frac{1}{s+3},$$

we can evaluate input signal in s-domain as

$$X(s) = \frac{Y(s)}{H(s)} = \frac{(s+3)^2+4^2}{(s+3)^2} = 1 + \frac{16}{(s+3)^2}.$$

After inverse transform, we get

$$x(t) = \delta(t) + 16r(t)e^{-3t}u(t).$$

Question 3

Find the Laplace transforms and ROC of the following signals

(i) $y(t) = e^{-2t}\cos^2(t - \pi/3)u(t)$

(ii) $y(t) = \int_0^t \sin(t-\tau)e^{-(2t+3\tau)}\cos(\tau)d\tau$.

Solution:

(i) Using trigonometric identity $\cos^2(\theta) = 0.5[\cos(2\theta) + 1]$

$$y(t) = 0.5\cos(2t - 2\pi/3)e^{-2t}u(t) + 0.5e^{-2t}u(t)$$

Inverse tranform of the first term can be evaluated by writing it as

$$2Ae^{-at}\cos(\Omega_0 t + \theta)u(t)$$

with $A = 1/4, a = 2, \Omega_0 = 2, \theta = 2\pi/3$. Therefore

$$Y(s) = \frac{1}{4}\left(\frac{e^{-j2\pi/3}}{s+2-2j} + \frac{e^{j2\pi/3}}{s+2+2j}\right) + \frac{1}{2(s+2)},$$

which can be further simplified to

$$\begin{aligned} Y(s) =& \frac{\cos(2\pi/3)(s+2) + 2\sin(2\pi/3)}{2(s^2+4s+8)} + \frac{1}{2(s+2)} \\ =& \frac{-s-2+2\sqrt{3}}{4(s^2+4s+8)} + \frac{1}{2(s+2)} \end{aligned}$$

The ROC is $\mathcal{R}e(s) > -2$.

(ii) The time domain signal can be written as convolution of two signals as

$$\begin{aligned} y(t) =& \int_{-\infty}^{\infty} \sin(t-\tau)e^{-2(t-\tau)}u(t-\tau)e^{-5\tau}\cos(\tau)u(\tau)d\tau. \\ =& x(t) * h(t). \end{aligned}$$

where

$$x(t) = \sin(t)e^{-2t}u(t), \quad h(t) = \cos(t)e^{-5t}u(t)$$

Therefore, by finding the Laplace Transform of $x(t)$ and $h(t)$

$$\begin{aligned} X(s) =& \frac{1}{(s+2)^2+1} \text{ or } X(s) = \frac{1}{s^2+4s+5}. \\ H(s) =& \frac{s+5}{(s+5)^2+1} \text{ or } H(s) = \frac{s+5}{s^2+10s+26}. \end{aligned}$$

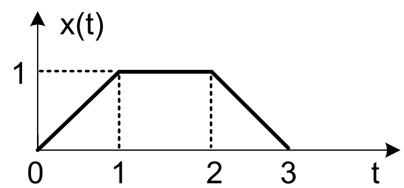
we can evaluate the Laplace Transform of $y(t)$ as

$$Y(s) = X(s)H(s) = \frac{1}{(s+2)^2+1}\frac{s+5}{(s+5)^2+1}.$$

It has two pair of poles, $s = -2 \pm 1j$ and $s = -5 \pm 1j$. Since $y(t)$ is causal, ROC is $\mathcal{R}e(s) > -2$.

Question 4

The following figure shows a time domain signal $x(t)$



(i) Sketch signal

$$y(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - 6n)$$

(* denotes convolution)

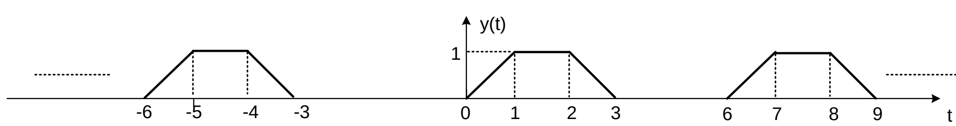
(ii) Compute the Fourier series coefficients Y_k of $y(t)$ by integral definition of Fourier series.

(iii) Compute the Fourier series coefficients Y_k using Laplace transform of $x(t)$.

Solution

(i)

$y(t)$ repeats periodically with period $T_0 = 6$.



(ii) The period and fundamental frequency of $y(t)$ are

$$T_0 = 6, \quad w_0 = 2\pi/T_0 = \pi/3.$$

Using definition of Fourier series coefficients:

$$\begin{aligned}
Y_0 &= \frac{1}{T_0}\int_0^6 y(t)dt = 1/3, \\
Y_k &= \frac{1}{T_0}\int_0^6 y(t)e^{-jk\frac{\pi}{3}t}dt \\
&= \frac{1}{T_0}\left[\int_0^1 te^{-jk\frac{\pi}{3}t}dt + \int_1^2 e^{-jk\frac{\pi}{3}t}dt + \int_2^3 (3-t)e^{-jk\frac{\pi}{3}t}dt\right].
\end{aligned}$$

Solve using integration by parts to obtain

$$Y_k = \frac{1}{6}\left[\frac{-e^{-j3k\pi/3} + e^{-j2k\pi/3} + e^{-jk\pi/3} - 1}{(k\pi/3)^2}\right] \tag{1}$$

(iii)
The period and fundamental frequency of $y(t)$ are

$$T_0 = 6, \quad w_0 = 2\pi/T_0 = \pi/3.$$

Further, $x(t) = r(t) - r(t-1) - r(t-2) + r(t+3)$. Therefore,

$$X(s) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s}). \tag{2}$$

Fourier coefficients are

$$\begin{aligned}
Y_k =& \frac{X(s = jkw_0)}{T_0} \\
=& \frac{1}{6}\left[\frac{-e^{-j3k\pi/3} + e^{-j2k\pi/3} + e^{-jk\pi/3} - 1}{(k\pi/3)^2}\right] \\
=& \frac{3}{2}\left[\frac{(-1)^{k+1} - 1 + e^{-j2k\pi/3} + e^{-jk\pi/3}}{k^2\pi^2}\right], k \neq 0.
\end{aligned}$$

Substituting $k = 0$ in the above equation yields $0/0$. Using L'Hospital's rule and differentiating the denominator and numerator twice with respect to k and substituting $k = 0$ gives $Y_0 = 1/3$, i.e.

$$Y_0 = \frac{1}{6}\left[\frac{\frac{d^2}{dk^2}\left(-e^{-j3k\pi/3} + e^{-j2k\pi/3} + e^{-jk\pi/3} - 1\right)}{\frac{d^2}{dk^2}(k\pi/3)^2}\right]_{k=0} = \frac{1}{3}.$$

Question 5

Find a periodic signal $x(t)$ with period $T_0 = 4$. The following information is given:

($X_k's$ are Fourier series coefficients of $x(t)$)

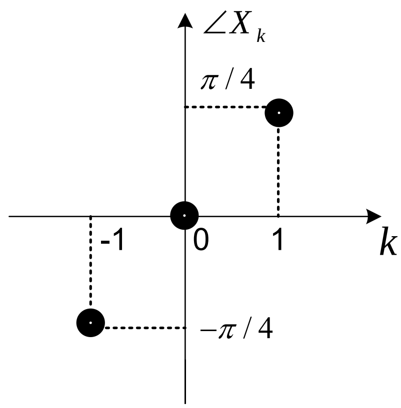
(a) $|X_k| = |X_{-k}|$.

(b) $X_k = 0, \text{for } |k| \geq 2$.

(c) $\frac{1}{4}\int_0^4 x(t)dt = 1$.

(d) $\frac{1}{4}\int_0^4 |x(t)|^2 dt = 33$.

(e) The phase of X_k is shown in the figure below:



Solution:

The period and fundamental frequency are

$$T_0 = 4, \quad w_0 = \pi/2$$

We can use fourier series to represent signal $x(t)$ by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkw_0 t}$$

Given $X_k = 0, \forall |k| \geq 2$, the above expression can be reduced to

$$x(t) = X_{-1}e^{-j\omega_0 t} + X_0 + X_1 e^{j\omega_0 t}$$

$X_0 = 1 = \frac{1}{4}\int_0^4 x(t)dt$
using Parseval's theorem, and $X_{-1} = X_1$
$1 + 2|X_1|^2 = 33 = \frac{1}{4}\int_0^4 |x(t)|^2 dt$
$|X_1| = 4$

Using the phase of X_k, $X_1 = 4e^{j\pi/4}, X_{-1} = 4e^{-j\pi/4}$ and

$$x(t) = 4e^{-j\pi/4}e^{-j\frac{\pi}{2}t} + 1 + 4e^{j\pi/4}e^{j\frac{\pi}{2}t} \tag{3}$$

$$x(t) = 1 + 8\cos\left(\frac{\pi}{2}t + \pi/4\right) \tag{4}$$

Question 6

Eigen-function property in frequency domain:
Consider an LTI system with transfer function $H(s)$. If input $x(t) = \cos(\omega_0 t)$ is applied to this system, the corresponding output is $y(t) = j\sin(\omega_0 t)H(j\omega_0)$. ($\omega_0$ is a constant.)
(i) Find the condition on $H(j\omega_0)$ that results in given input-output pair.
(ii) Find the output $y_2(t)$ of system if input is $x_2(t) = j\sin(\omega_0 t)$. Use condition in (i) to express $y_2(t)$ in terms of $\cos(\omega_0 t)$.
Solution:

(i)

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Using the eigenfunction property, the corresponding output is

$$y'(t) = \frac{e^{j\omega_0 t}H(j\omega_0) + e^{-j\omega_0 t}H(-j\omega_0)}{2}$$

But, the given output is:

$$y(t) = j\sin(\omega_0 t)H(j\omega_0) = \frac{e^{j\omega_0 t}H(j\omega_0) - e^{-j\omega_0 t}H(j\omega_0)}{2}$$

Therefore, the required condition is $H(j\omega_0) = -H(-j\omega_0)$.
(ii) Using the eigenfunction property, output corresponding to input $x_2(t) = j\sin(\omega_0 t)$ is

$$y_2(t) = \frac{e^{j\omega_0 t}H(j\omega_0) - e^{-j\omega_0 t}H(-j\omega_0)}{2}$$

Using the condition $H(j\omega_0) = -H(-j\omega_0)$, the output reduces to

$$y_2(t) = \frac{e^{j\omega_0 t}H(j\omega_0) + e^{-j\omega_0 t}H(j\omega_0)}{2} = \cos(\omega_0 t)H(j\omega_0)$$