EE102 Systems and Signals

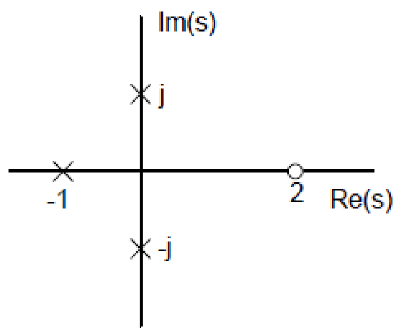
Practice Questions for Laplace Transform

Question 1

The figure below is the "poles-zeros" plot of a system function $H(s)$:

(i) (5 pt) Find $H(s)$ given $H(0) = \sqrt{2}$.

(ii) (10 pt) Find the output of the system when $tu(t-1)$ is applied, knowing that the system was at rest.



Question 2

Given $x(t) = \sqrt{2}e^{-t}u(t)$ and $y(t) = \sqrt{2}e^{-t}u(t)(1 - 2t)$, find the impulse response function $h(t)$ of the linear time-invariant system which admits the given input-output pair.

Question 3

The system function $H(s)$ of a linear time-invariant systems S is

$$H(s) = \frac{1}{s^2 + s + 1}.$$

(i) (2 pt) If $x(t)$ and $y(t)$ are the input and output of S, respectively, find the differential equation relating $x(t)$ and $y(t)$. Assume the system is at rest.

(ii) (10 pt) Find $y(t)$ when $x(t) = \sin(2(t-1))u(t-1)$.

(iii) (5 pt) Find the step response $g(t)$ of S–by direct calculation in the time domain and by Laplace transform.

Question 4

Compute

$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)u(t-\sigma)x(\sigma)d\sigma,$$

for

(i) (5 pt) $h(t) = t^2 u(t)$, $x(t) = e^{-t}u(t-5)$.
(ii) (5 pt) $h(t) = \sin(\omega t)$, $x(t) = tu(t)$.

Q1 15 (i)

5

$$H(s) = \frac{K.(s-2)}{(s-j)(s+j)(s+1)} \qquad k = \text{const.}$$

$$= \frac{K.(s-2)}{(s^2+1)(s+1)}.$$

$$H(0) = \frac{K.(-2)}{1} = \sqrt{2}. \Rightarrow K = \frac{-1}{\sqrt{2}}.$$

$$\therefore \boxed{H(s) = \frac{-1}{\sqrt{2}} \frac{(s-2)}{(s^2+1)(s+1)}}.$$

(ii) 10

$$x(t) = t\, u(t-1)$$

$$= (t-1)\,u(t-1) + u(t-1)$$

$$\therefore X(s) = e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$= \frac{e^{-s}}{s^2}\left[s+1\right]$$

$$Y(s) = H(s) \cdot X(s)$$

$$= \frac{-1}{\sqrt{2}} \frac{(s-2)}{(s^2+1)\cancel{(s+1)}} \frac{e^{-s}\cancel{(s+1)}}{s^2}.$$

$$= \frac{-e^{-s}}{\sqrt{2}} \left(\frac{s-2}{(s^2+1)\,s^2}\right] \quad \text{—} (1)$$

Let $Y_1(s) = \dfrac{s-2}{s^2(s^2+1)} = \dfrac{As+B}{s^2} + \dfrac{Cs+D}{s^2+1}$

* Comparing coefficients of equal powers of s on both sides.

for s^3: $A+C=0$

s^2: $B+D=0$

s : $A=1$

const : $B=-2$.

$\Rightarrow D=2,\ C=-1$

$$\therefore Y_1(s) = \frac{s-2}{s^2} - \frac{s-2}{s^2+1}$$

$$= \frac{1}{s} - \frac{2}{s^2} - \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

$$y_1(t) = u(t) - 2t\,u(t) - \cos(t)u(t) + 2\sin(t)u(t)$$

From ①. $Y(s) = -\dfrac{e^{-s}}{\sqrt{2}} Y_1(s)$.

$$\therefore y(t) = \frac{-1}{\sqrt{2}} y_1(t-1)$$

$$= \frac{-1}{\sqrt{2}}\Big[u(t-1) - 2(t-1)u(t-1) - \cos(t-1)u(t-1) + 2\sin(t-1)u(t-1)\Big]$$

$3-2t$

Q2

$$X(s) = \frac{\sqrt{2}}{s+1}$$

$$Y(s) = \frac{\sqrt{2}}{s+1} - \frac{2\sqrt{2}}{(s+1)^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{\sqrt{2}(s+1) - 2\sqrt{2}}{(s+1)^2}}{\frac{\sqrt{2}}{s+1}} = \frac{(s+1)-2}{s+1} = \frac{s-1}{s+1}$$

$$= \frac{s}{s+1} - \frac{1}{s+1} = 1 - \frac{2}{s+1}$$

$$\Rightarrow \quad h(t) = \cancel{\cos t\, u(t) - \sin t\, u(t)}$$

$$= \delta(t) - 2e^{-t}u(t)$$

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Q3

(i) $$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+s+1}$$

$$\Rightarrow \quad s^2Y(s) + sY(s) + Y(s) = X(s)$$

$$\Rightarrow \quad \frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

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(ii) $x(t) = \sin(2(t-1))\,u(t-1)$

$$\Rightarrow X(s) = \frac{2e^{-s}}{s^2+4}$$

$$\Rightarrow Y(s) = H(s)X(s) = \frac{1}{s^2+s+1} \cdot \frac{2e^{-s}}{s^2+4}$$

$$= \left[\frac{2}{(s^2+4)(s^2+s+1)}\right] e^{-s}$$

$$= G(s)\, e^{-s}$$

$$G(s) = \frac{2}{(s^2+4)(s^2+s+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+s+1}$$

$$\Rightarrow \quad A = \frac{-2}{13},\ B = \frac{-6}{13},\ C = \frac{2}{13},\ D = \frac{8}{13}.$$

$$\Rightarrow G(s) = A\frac{s}{s^2+4} + B\cdot\frac{1}{s^2+4} + \frac{C(s+\frac{1}{2}) + D}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow g(t) = A\cos(2t)u(t) + \frac{B}{2}\sin(2t)u(t) + Ce^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t) + \frac{2D}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

$$\Rightarrow \quad y(t) = g(t-1)$$ ※

(iii) ① Direct Calculation.

$$H(s) = \frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \Rightarrow h(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) u(t)$$

$$\Rightarrow g(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-\tau)} \sin\left(\frac{\sqrt{3}}{2}(t-\tau)\right) u(t-\tau)\, u(\tau)\, d\tau$$

$$= \int_0^t \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} e^{\frac{1}{2}\tau} \sin\left(\frac{\sqrt{3}}{2}(t-\tau)\right) d\tau$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \int_0^t e^{\frac{1}{2}\tau} \left[\frac{e^{j[\frac{\sqrt{3}}{2}(t-\tau)]} - e^{-j[\frac{\sqrt{3}}{2}(t-\tau)]}}{2j} \right] d\tau$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \int_0^t \frac{1}{2j} \left[e^{j\frac{\sqrt{3}}{2}t} \cdot e^{(\frac{1}{2}-\frac{\sqrt{3}}{2}j)\tau} - e^{-j\frac{\sqrt{3}}{2}t} \cdot e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}j)\tau} \right] d\tau$$

$$= \frac{\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t}}{2j} e^{j\frac{\sqrt{3}}{2}t} \left[\frac{e^{(\frac{1}{2}-\frac{\sqrt{3}}{2}j)t} - 1}{\frac{1}{2}-\frac{\sqrt{3}}{2}j} \right] - \frac{\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t}}{2j} e^{-j\frac{\sqrt{3}}{2}t} \left[\frac{e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}j)t} - 1}{\frac{1}{2}+\frac{\sqrt{3}}{2}j} \right]$$

$$= 1 - e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right) \#$$

② By Laplace

$$X(s) = \frac{1}{s} \Rightarrow G(s) = H(s)X(s) = \frac{1}{s(s^2+s+1)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$\Rightarrow A = 1,\ B = -1,\ C = -1.$$

$$\Rightarrow G(s) = \frac{1}{s} + \frac{-s-1}{s^2+s+1} = \frac{1}{s} - \frac{s+1}{s^2+s+1}$$

$$G(s) = \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right) + \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad g(t) = u(t) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

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Q4

$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)\,u(t-\sigma)\,x(\sigma)\,d\sigma$$

$$= h(t)u(t) * x(t)$$

(i)

$$h(t) = t^2 u(t), \quad x(t) = e^{-t}u(t-5)$$

$$\Rightarrow H(s) = -\frac{d\frac{1}{s^2}}{ds} = \frac{2}{s^3}, \quad X(s) = e^{-5}\cdot\frac{e^{-5s}}{s+1}$$

$$\Rightarrow \quad Y(s) = H(s)\cdot X(s) = \frac{2}{s^3}\cdot\frac{e^{-5s}\cdot e^{-5}}{s+1}$$

$$= e^{-5s}\cdot 2e^{-5}\left[\frac{1}{(s+1)s^3}\right]$$

$$\frac{1}{(s+1)s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}, \quad \Rightarrow \quad C=1,\ B=-1,\ A=1,\ D=-1$$

$$= \frac{1}{s} + \frac{-1}{s^2} + \frac{1}{s^3} + \frac{-1}{s+1}$$

$$\Rightarrow \quad y(t) = 2e^{-5}\left[u(t-5) - (t-5)u(t-5) + \frac{1}{2}(t-5)^2 u(t-5) - e^{-(t-5)}u(t-5)\right]$$

(ii) $h(t) = \sin(wt)$ $\quad x(t) = t u(t)$

$$H(s) = \frac{w}{s^2+w^2} \qquad X(s) = \frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{w}{s^2(s^2+w^2)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+w^2}$$

$\Rightarrow$ $A = 0$, $B = \frac{1}{w}$, $C = 0$, $D = \frac{-1}{w}$

$$\Rightarrow Y(s) = \frac{1}{w} \cdot \frac{1}{s^2} + \frac{\frac{-1}{w}}{s^2+w^2}$$

$$\Rightarrow y(t) = \frac{1}{w} t u(t) - \frac{1}{w^2} \sin(wt) u(t)$$

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