UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm Examination
February 20, 2019
Duration: 1 hr 50 mins.

INSTRUCTIONS:

- The exam has 6 problems and 16 pages.
- The exam is closed-book.
- One cheat sheet of A4 size is allowed.
- Calculator is NOT allowed.
- Write your discussion session in the top-right corner. ↗↗

Your name:– ______________

Student ID:– ______________

Table 1: Score Table

Problem	a	b	c	d	e	Score	
1	5	5				10	10
2	2	2	2	5	2	13	8
3	5	2	5			12	12
4	10					10	10
5	5	5	5			15	13
6	7	8				15	15
Total						75	68

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t)$, $a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t)$, $a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t)$, $a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t)$, $a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\, t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\, t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\, t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t)$, $\beta g(t)$	$\alpha F(s)$, $\beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)$ $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

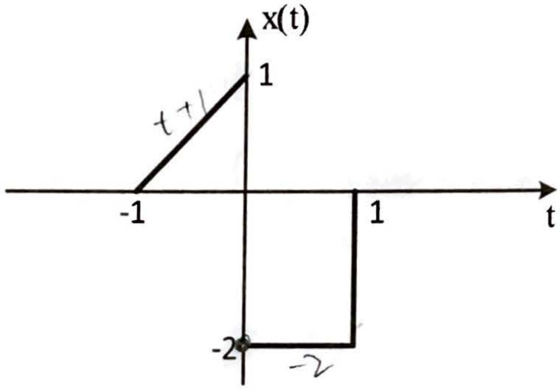
Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Problem 1 (10 pts)

Consider signal $x(t)$ depicted in the figure below



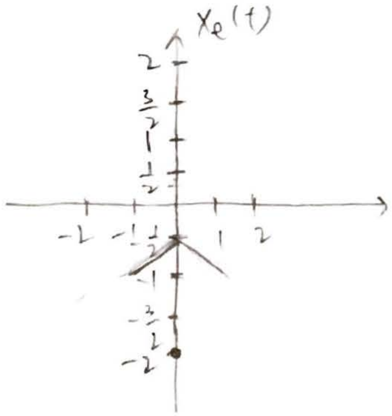
(a) (5 pts) Sketch even and odd components of $x(t)$. Assume $x(0) = -2$.

(b) (5 pts) Sketch $2x(-2t-1)$.

a) $x(t) = \begin{cases} t+1 & -1 \le t < 0 \\ -2 & 0 \le t \le 1 \end{cases}$ $\qquad x(-t) = \begin{cases} -t+1 & 0 < t \le 1 \\ -2 & -1 \le t \le 0 \end{cases}$

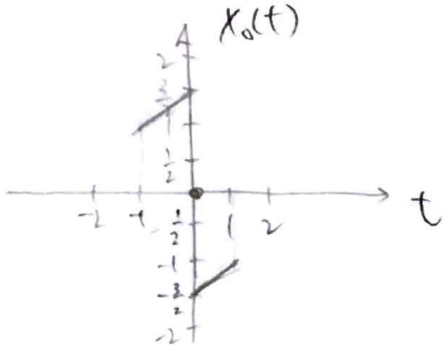
$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}\begin{cases} t+1-2 & -1 \le t < 0 \\ -2-t+1 & 0 < t \le 1 \\ -2-2 & t = 0 \end{cases}$$

$$x_e(t) = \frac{1}{2}\begin{cases} t-1 & -1 \le t < 0 \\ -t-1 & 0 < t \le 1 \\ -4 & t = 0 \end{cases} = \begin{cases} \frac{1}{2}t - \frac{1}{2} & -1 \le t < 0 \\ -\frac{1}{2}t - \frac{1}{2} & 0 < t \le 1 \\ -2 & t = 0 \end{cases}$$



$$x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] = \frac{1}{2}\begin{cases} t+1+2 & -1 \le t < 0 \\ -2+t-1 & 0 < t \le 1 \\ 0 & t = 0 \end{cases}$$

$$= \frac{1}{2}\begin{cases} t+3 & -1 \le t < 0 \\ t-3 & 0 < t \le 1 \\ 0 & t = 0 \end{cases} = \begin{cases} \frac{1}{2}t + \frac{3}{2} & -1 \le t < 0 \\ \frac{1}{2}t - \frac{3}{2} & 0 < t \le 1 \\ 0 & t = 0 \end{cases}$$



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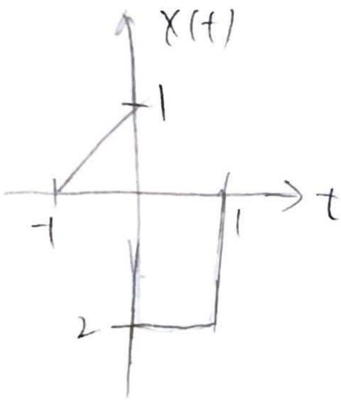
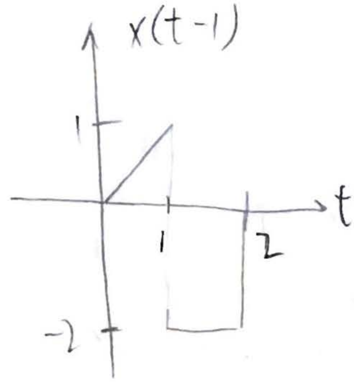
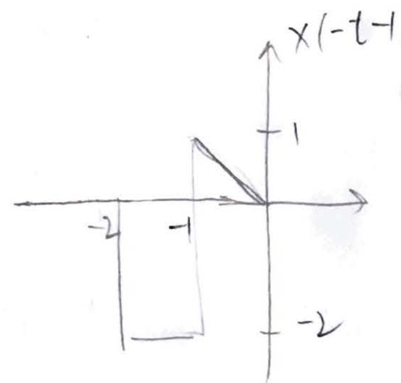
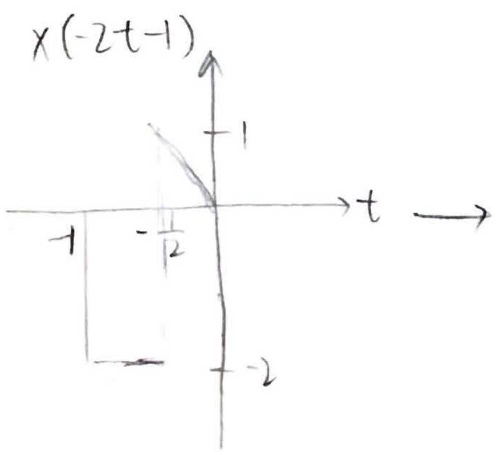
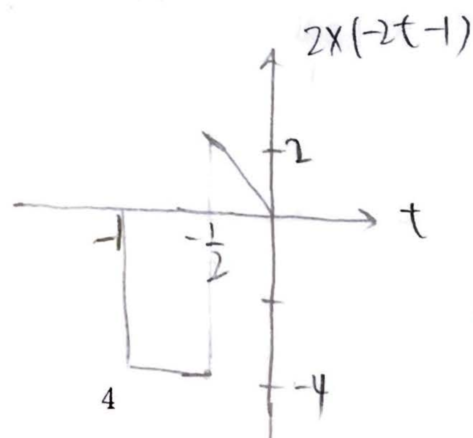
b) $2x(-2t-1)$











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Problem 2 (13 pts)

The system S is given by the following relation

$$y(t) = x(t-2) + x(2-t), \quad -\infty < t < \infty$$

1) (6 pts) Using the system relation, answer the following questions about the system. You need to justify your answer.

   (a) (2 pts) Prove that the system is linear.

   (b) (2 pts) Is the system time-varying or time-invariant?

   (c) (2 pts) Is the system causal or not causal?

2) (5 pts) Find the impulse response function of the system.

3) (2 pts) Verify your answers for parts b) and c) in 1) using the impulse response function of the system. You need to justify your answer.

1a) $S[\alpha v(t) + \beta w(t)] \stackrel{L}{=} \alpha S[v(t)] + \beta S[w(t)]$

$S[x(t)] = x(t-2) + x(2-t)$

$S[\alpha v(t) + \beta w(t)] = (\alpha v(t-2) + \beta w(t-2)) + (\alpha v(2-t) + \beta w(2-t))$

$\alpha S[v(t)] = \alpha[v(t-2) + v(2-t)] = \alpha v(t-2) + \alpha v(2-t)$

$\beta S[w(t)] = \beta[w(t-2) + w(2-t)] = \beta w(t-2) + \beta w(2-t)$

$(\alpha v(t-2) + \beta w(t-2)) + (\alpha v(2-t) + \beta w(2-t)) \stackrel{L}{=} \alpha v(t-2) + \alpha v(2-t) + \beta w(t-2) + \beta w(2-t)$

system is linear b/c inputing a superposition of inputs results in the same output as when adding up all the outputs for those inputs separately

2

1b) $S[x(t-\tau)] \stackrel{TI}{=} y(t-\tau)$

$y(t) = x(t-2) + x(2-t)$

$y(t-\tau) = x(t-\tau-2) + x(2-(t-\tau)) = x(t-\tau-2) + x(2-t+\tau)$

$S[x(t-\tau)] = x(t-\tau-2) + x(2-(t-\tau)) = x(t-\tau-2) + x(2-t+\tau)$

$S[x(t-\tau)] \stackrel{TI}{=} y(t-\tau)$ ✓

1 system is <u>time invariant</u> because shifting the input produces the same output shifted by a certain amount.

1c) causal; $y(t)$ does not depend on anything that is later than t. (depends on $t-2$, which is not later than t)

0

2. $y(t) = x(t-2) + x(-(t-2))$

$x(t) = \delta(t-\tau)$

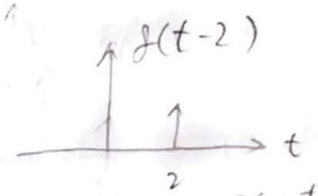
$h(t,\tau) = \delta(t-\tau-2) + \delta(-(t-\tau-2))$

$h(t) = \delta(t-2) + \delta(-(t-2))$

3

3) The impulse response $h(t,\tau)$ shows the system is <u>time-invariant</u> because $h(t,\tau) = h(t-\tau)$. $h(t;\tau)$ depends on only $(t-\tau)$, so it can be rewritten as $h(t) = \delta(t-2) + \delta(-(t-2))$, where $h(t)$ only depends on 1 variable

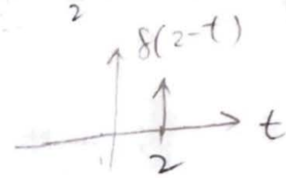
$$h(t) = \delta(t-2) + \delta(-(t-2))$$



does $h(t) = 0$ when $t<0$?

2

$$h(\text{neg}\#) = \underbrace{\delta(\text{neg}\# - 2)}_{\delta(\text{neg}\#) = 0} + \underbrace{\delta(2 - \text{neg}\#)}_{\delta(\text{pos}\#) = 0}$$



<u>causal</u> b/c $h(t) = 0$ when $t<0$

Problem 3 (12 pts)

Consider an LTI system with the following input-output relationship:

$$y(t) = \int_{-\infty}^{t-1} e^t cos(2\tau + 2 - 2t)x(\tau)e^{-\tau+2}d\tau$$

where $x(t)$ and $y(t)$ are the input and the output of the system, respectively.

(a) (5 pts) Find the impulse response function $h(t)$.

(b) (2 pts) Is the system C or NC? Provide justification.

(c) (5 pts) Is this system BIBO stable? Provide justification.

a) $y(t) = \int_{-\infty}^{t-1} e^t \cos(-2(t-\tau+1))\, x(\tau)\, e^{-(\tau-2)}\, d\tau$

$y(t) = \int_{-\infty}^{+\infty} e^{t-\tau+2} \cos(-2(t-\tau)+2)\, u(t-1-\tau)\, x(\tau)\, d\tau$

$h(t,\tau) = e^{t-\tau+2} \cos(-2(t-\tau)+2)\, u(t-\tau-1)$

$\boxed{h(t) = e^{t+2} \cos(2-2t)\, u(t-1)}$ ✓5

b) $h(t)=0$ when $t<0$?

$h(\text{neg}\#) = e^{(\#)} \cos(2-2(\text{neg}\#))\, u(\text{neg}\# - 1)$

system is causal because $h(t)=0$ when $t<0$

because of the unit step $u(t-1)$. ✓2

c) $\int_{-\infty}^{+\infty} |h(t)|\, dt < +\infty$ to be BIBO stable

$h(t) = e^{t+2} \cos(2-2t)\, u(t-1)$

$\int_{-\infty}^{+\infty} \left| e^{t+2} \cos(2-2t)\, u(t-1) \right| dt$

$\int_{-\infty}^{+\infty} u(t-1) \left| e^{2} e^{t} \underbrace{\cos(2-2t)}_{=|\pm 1|} \right| dt$

$e^{2} \int_{1}^{+\infty} |e^{t}|\, dt = e^{2} \left[|e^{t}| \Big|_{1}^{+\infty} \right] \longrightarrow \infty$

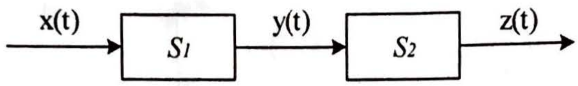
System is <u>NOT</u> BIBO stable b/c

$\int_{-\infty}^{+\infty} |h(t)|\, dt \longrightarrow \infty$ (not a finite #)

$\sqrt{}$ 5

Problem 4 (10 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.



The first system S_1 is described by:

$$y(t) = \int_0^t e^{-\sigma} t x(\sigma) d\sigma, \quad t > 0$$

where $x(t)$ and $y(t)$ are the input and the output, respectively. The second system is described by:

$$z(t) = \int_2^t \sigma y(\sigma) d\sigma, \quad t > 2$$

where $y(t)$ and $z(t)$ and the input and the output, respectively.

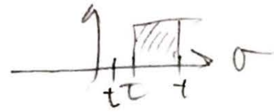
Find the impulse response function $h_{12}(t, \tau)$ of the cascaded system S_{12}.

$$h_1(t,\tau) = \int_0^t e^{-\sigma} t\, \delta(\sigma-\tau)\, d\sigma = te^{-\tau} \int_0^t \delta(\sigma-\tau)\, d\tau$$

$$x(t) = \delta(t-\tau) \qquad = te^{-\tau} \begin{cases} 0 & 0 \le t < \tau \\ 1 & t \ge \tau \end{cases}$$

$$\underline{h_1(t,\tau) = te^{-\tau} u(\tau) u(t-\tau)} \quad \checkmark_4 \qquad y(t) = h_1(t,\tau)$$

$$h_{12}(t,\tau) = \int_2^t \sigma\left(\sigma e^{-\tau} u(\tau) u(\sigma-\tau)\right) d\sigma = e^{-\tau} u(\tau) \int_2^t \sigma^2 u(\sigma-\tau)\, d\sigma$$



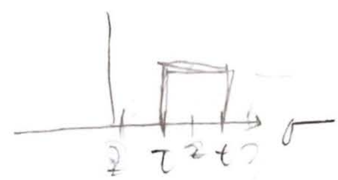
$$\cancel{= e^{-\tau} u(\tau) \begin{cases} 0 & 2 \le t < \tau \\ \int_\tau^t \sigma^2 d\sigma & t \ge \tau \end{cases}}$$

$$h_{12}(t,\tau) = e^{-\tau}u(\tau)\begin{cases} 0 & 2 \le t < \tau \\ \frac{1}{3}\sigma^3\Big|_\tau^t & t \ge \tau \end{cases}$$

$$= e^{-\tau}u(\tau)\begin{cases} 0 & 2 \le t < \tau \\ \frac{1}{3}t^3 - \frac{1}{3}\tau^3 & t \ge \tau \ge 2 \end{cases}$$

$$\boxed{h_{12}(t,\tau) = e^{-\tau}u(\tau)\,\frac{1}{3}(t^3-\tau^3)\,u(t-\tau)\,u(t-2)}$$

$$h_{12}(t,\tau) = e^{-\tau}u(\tau)\int_\tau^t \sigma^2 u(\sigma-2)\,d\sigma$$



$$= e^{-\tau}u(\tau)\begin{cases} 0 & \cancel{\tau < t < 2}\ \ \tau \ge t \\ \frac{1}{3}\sigma^3\Big|_\tau^t & t > \tau > 2 \\ \frac{1}{3}\sigma^3\Big|_2^t & \tau \le 2 \end{cases}$$

$$\int \sigma^2 d\sigma = \frac{1}{3}\sigma^3$$

$$\boxed{h_{12}(t,\tau) = e^{-\tau}u(\tau)\begin{cases} 0 & \cancel{\tau < t < 2}\ \ \tau \ge t \\ \frac{1}{3}t^3 - \frac{1}{3}\tau^3 & 2 < \tau < t \\ \frac{1}{3}t^3 - \frac{8}{3} & \tau \le 2 \end{cases}}$$

√6

10

13

Problem 5 (15 pts)

1) (5 pts) Find the Laplace transform and ROC of the following function

$$f(t) = te^{-3t}(sin(t) + cos(t))^2 u(t).$$

Hint: $sin(2x) = 2sin(x)cos(x)$.

2) (10 pts) Find the functions that correspond to the following Laplace transforms. Assume that time-domain functions are causal.

   (a) (5 pts) $F(s) = \frac{s}{(s+1)^2+3}$

   (b) (5 pts) $F(s) = \frac{1}{s^4}(1 - e^{-s})^2$.

1) $f(t) = t e^{-3t}(\sin^2(t) + \underbrace{2\sin t \cos t}_{\sin(2t)} + \cos^2 t)\, u(t)$

3

$\sin^2 t + \cos^2 t = 1$

$f(t) = te^{-3t}(\sin^2(t) + \sin(2t) + \cos^2 t)\, u(t)$

$f(t) = te^{-3t}\left(\frac{1}{2} - \cancel{\frac{1}{2}\sin(2t)} + \sin(2t) + \frac{1}{2} + \cancel{\frac{1}{2}\cos(2t)}\right) u(t)$

$f(t) = te^{-3t}\left(1 + \frac{1}{2}\sin(2t) + \frac{1}{2}\cos(2t)\right) u(t)$

$f(t) = e^{-3t} t u(t) + \frac{1}{2} e^{-3t} t \sin(2t) u(t) + \frac{1}{2} e^{-3t} t \cos(2t) u(t)$

$e^{-3t} t u(t) \xrightarrow{LS} \frac{1}{(s+3)^2}$

$\sin(2t) u(t) \xrightarrow{LS} \frac{2}{s^2+4}$

$\cos(2t) u(t) \xrightarrow{LS} \frac{s}{s^2+4}$

$t\sin(2t) u(t) \xrightarrow{LS} -\frac{d}{ds}\left(2(s^2+4)^{-1}\right)$

$-\frac{d}{ds}\left(-2(s^2+4)^{-2}(2s)\right)$

$\frac{4s}{(s^2+4)^2}$

$t\cos(2t) u(t) \xrightarrow{LS} -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$-\left(\frac{(s^2+4) - s(2s)}{(s^2+4)^2}\right) = -\frac{s^2+4-2s^2}{(s^2+4)^2} \quad \frac{s^2-4}{(s^2+4)^2}$

$\frac{1}{2} e^{-3t} t \cos(2t) u(t) \xrightarrow{LS} \frac{1}{2} \frac{(s+3)^2 - 4}{((s+3)^2+4)^2}$

$\frac{1}{2} e^{-3t} t \sin(2t) u(t) \xrightarrow{LS} \frac{1}{2} \frac{4(s+3)}{((s+3)^2+4)^2}$

$$\boxed{F(s) = \frac{1}{(s+3)^2} + \frac{1}{2}\,\frac{4(s+3)}{((s+3)^2+4)^2} + \frac{1}{2}\,\frac{(s+3)^2-4}{((s+3)^2+4)^2}}$$

2 a) $F(s) = \dfrac{(s+1)-1}{(s+1)^2+3} = \dfrac{s+1}{(s+1)^2+(\sqrt{3})^2} - \dfrac{1\left(\frac{\sqrt{3}}{\sqrt{3}}\right)}{(s+1)^2+(\sqrt{3})^2}$

$$\boxed{f(t) = e^{-t}\cos(\sqrt{3}t)u(t) - \frac{1}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)u(t)}$$

5

2 b) $F(s) = \frac{1}{s^4}(1-e^{-s})^2 = \frac{1}{s^4}(1-2e^{-s}+e^{-2s}) = \frac{1}{s^4} - 2e^{-s}\frac{1}{s^4} + e^{-2s}\frac{1}{s^4}$

$$t^3u(t) \xrightarrow{\mathcal{L}} \frac{3!}{s^4} = \frac{6}{s^4} \qquad \frac{1}{6}t^3u(t) \xrightarrow{\mathcal{L}} \frac{1}{s^4}$$

$f(t) = \frac{1}{6}t^3u(t) - \frac{2}{6}(t-1)^3u(t-1) + \frac{1}{6}(t-2)^3u(t-2)$

$$\boxed{f(t) = \frac{1}{6}t^3u(t) - \frac{1}{3}(t-1)^3u(t-1) + \frac{1}{6}(t-2)^3u(t-2)}$$

5

Problem 6 (15 pts)

Consider an LTI system S with the following impulse response function

$$h(t) = (\frac{2}{3}e^{-t+1} + \frac{1}{3}e^{2t-2})u(t-1)$$

(a) (7 pts) Find the transfer function $H(s)$. Sketch its zero-pole plot and then denote ROC in the plot.

(b) (8 pts) If the output of the system is

$$y(t) = (-\frac{1}{5}cos(t-2) - \frac{2}{5}sin(t-2) + \frac{1}{5}e^{2(t-2)})u(t-2)$$

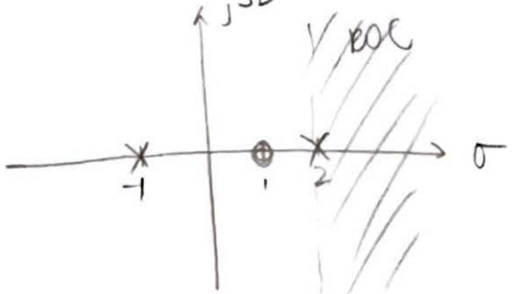
find its corresponding causal input $x(t)$.

a) $h(t) = \frac{2}{3} e^{-(t-1)} u(t-1) + \frac{1}{3} e^{2(t-1)} u(t-1)$

$$H(s) = \frac{2}{3} e^{-s} \frac{1}{s+1} + \frac{1}{3} e^{-s} \frac{1}{s-2} = e^{-s}\left(\frac{\frac{2}{3}(s-2) + \frac{1}{3}(s+1)}{(s+1)(s-2)}\right)$$

$$= e^{-s}\left(\frac{\frac{2}{3}s - \frac{4}{3} + \frac{1}{3}s + \frac{1}{3}}{(s+1)(s-2)}\right) = \boxed{e^{-s}\left(\frac{s-1}{(s+1)(s-2)}\right)}$$ √5

$z_1 = 1$, $p_1 = -1$, $p_2 = 2$



$\boxed{\text{ROC: } Re\{s\} > 2}$

√2

7

b) $y(t) = \left(-\frac{1}{5}\cos(t-2) - \frac{2}{5}\sin(t-2) + \frac{1}{5}e^{2(t-2)}\right)u(t-2)$

$y(t) = -\frac{1}{5}\cos(t-2)u(t-2) - \frac{2}{5}\sin(t-2)u(t-2) + \frac{1}{5}e^{2(t-2)}u(t-2)$

$Y(s) = -\frac{1}{5}\left(e^{-2s}\frac{s}{s^2+1}\right) - \frac{2}{5}\left(e^{-2s}\frac{1}{s^2+1}\right) + \frac{1}{5}e^{-2s}\frac{1}{s-2}$ ✓1

$Y(s) = -\frac{1}{5}e^{-2s}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{1}{s-2}\right] = -\frac{1}{5}e^{-2s}\left[\frac{\cancel{s^2}-2s+2s-4-\cancel{s^2}-1}{(s^2+1)(s-2)}\right]$

$= -\frac{1}{\cancel{5}}e^{-2s}\left[\frac{\cancel{-5}}{(s^2+1)(s-2)}\right] = e^{-2s}\frac{1}{(s^2+1)(s-2)}$

$H(s) = e^{-s}\frac{s-1}{(s+1)(s-2)}$ $Y(s) = H(s)X(s)$ ✓1

$e^{-2s}\frac{1}{(s^2+1)\cancel{(s-2)}} = e^{-s}\frac{s-1}{(s+1)\cancel{(s-2)}}X(s)$

$X(s) = e^{-s}\frac{s+1}{(s^2+1)(s-1)}$ ✓4 $= e^{-s}\left[\frac{As+B}{s^2+1} + \frac{C}{s-1}\right]$

$C = \left.\frac{s+1}{(s^2+1)\cancel{(s-1)}}\cancel{(s-1)}\right|_{s=1}$

$C = \frac{1+1}{1+1} = 1$

$= e^{-s}\left[\frac{-s}{s^2+1} + \frac{1}{s-1}\right]$

$(As+B)(s-1) + Cs^2 + C = s+1$

$As^2 - As + Bs - B + s^2 + 1 = s+1$

$(A+1)s^2 = 0$ $A = -1$

$-B+1 = 1$ $B = 0$

$-As + Bs = s$

$\boxed{X(t) = e^{(t-1)}u(t-1) - \cos(t-1)u(t-1)}$ ✓2

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