UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm Examination
February 20, 2019
Duration: 1 hr 50 mins.

INSTRUCTIONS:

- The exam has 6 problems and 16 pages.
- The exam is closed-book.
- One cheat sheet of A4 size is allowed.
- Calculator is NOT allowed.
- Write your discussion session in the top-right corner. ↗↗

Your name:____

Student ID:____

Table 1: Score Table

Problem	a	b	c	d	e	Score
1	5	5				10
2	2	2	2	5	2	13
3	5	2	5			12
4	10					10
5	5	5	5			15
6	7	8				15
Total						75

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

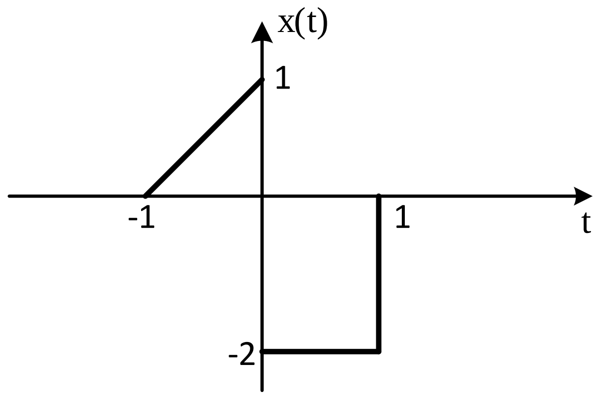
Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Problem 1 (10 pts)

Consider signal $x(t)$ depicted in the figure below



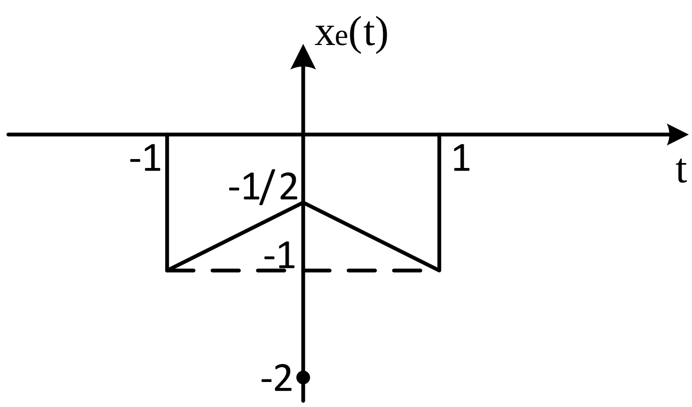
(a) (5 pts) Sketch even and odd components of $x(t)$. Assume $x(0) = -2$.

(b) (5 pts) Sketch $2x(-2t - 1)$.

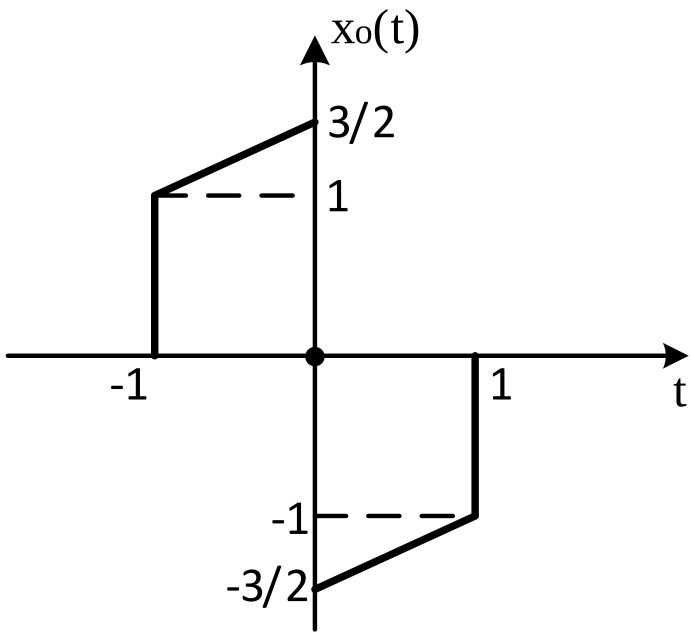
Solution:

(a)

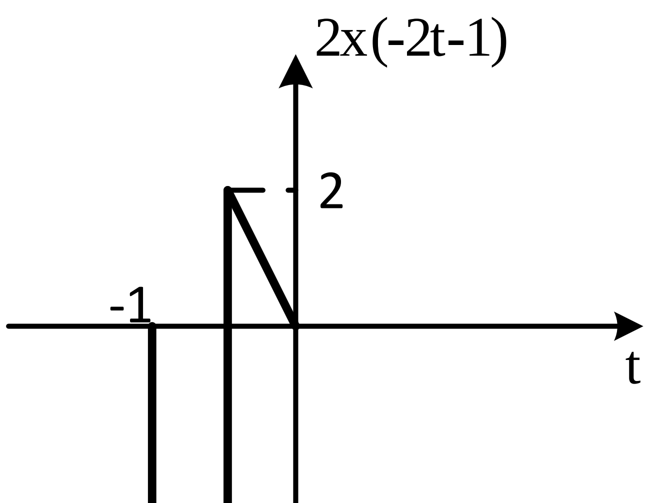
The even part of $x(t)$ can be found using $x_e = \frac{1}{2}(x(t) + x(-t))$.



The odd part of $x(t)$ can be found using $x_o = \frac{1}{2}(x(t) - x(-t))$.



(b)



Problem 2 (13 pts)

The system S is given by the following relation

$$y(t) = x(t-2) + x(2-t), \quad -\infty < t < \infty$$

1) (6 pts) Using the system relation, answer the following questions about the system. You need to justify your answer.
   (a) (2 pts) Prove that the system is linear.
   (b) (2 pts) Is the system time-varying or time-invariant?
   (c) (2 pts) Is the system causal or not causal?
2) (5 pts) Find the impulse response function of the system.
3) (2 pts) Verify your answers for parts b) and c) in 1) using the impulse response function of the system. You need to justify your answer.

Solution:

1)

(a)
Let $y_1(t)$ be the output of the system when the input is $x_1(t)$ and $y_2(t)$ be the output of the system when the input is $x_2(t)$ as follows

$$\begin{aligned} y_1(t) &= x_1(t-2) + x_1(2-t) \\ y_2(t) &= x_2(t-2) + x_2(2-t) \end{aligned}$$

Let $x_3(t) = \alpha x_1(t) + \beta y_2(t)$,

$$\begin{aligned} y_3(t) &= x_3(t-2) + x_3(2-t) \\ &= (\alpha x_1(t-2) + \beta x_2(t-2)) + (\alpha x_1(2-t) + \beta x_2(2-t)) \\ &= \alpha(x_1(t-2) + x_1(2-t)) + \beta(x_2(t-2) + x_2(2-t)) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

Hence, the system is linear.

(b)
Let $x_4(t) = x_1(t - t_0)$

$$\begin{aligned} y_4(t) &= x_4(t-2) + x_4(2-t) \\ &= x_1(t-2-t_0) + x_1(2-t-t_0) \end{aligned}$$

$$\begin{aligned} y_1(t-t_0) &= x_1(t-t_0-2) + x_1(2-(t-t_0)) \\ &= x_1(t-t_0-2) + x_1(2-t+t_0) \end{aligned}$$

$$y_4(t) \neq y_1(t-t_0)$$

Hence, the system is time varying.

(c)
If we take $t = -2$, we find that the value of y depends on the value of x at time -4 and 4. Since $4 > -2$, then this system is not causal.

2)

$$\begin{aligned} h(t,\tau) &= \delta(t-\tau-2) + \delta(2-t-\tau) \\ &= \delta(t-\tau-2) + \delta(2-t-\tau) \end{aligned}$$

3)

Since, $h(t,\tau) \neq h(t-\tau, 0)$, then the system is time varying. If we take $t = 0.5$, and $\tau = 1.5$, we find that $h(t,\tau)$is not equal to zero, hence the system is not causal.

Problem 3 (12 pts)

Consider an LTI system with the following input-output relationship:

$$y(t) = \int_{-\infty}^{t-1} e^t cos(2\tau + 2 - 2t)x(\tau)e^{-\tau+2}d\tau$$

where $x(t)$ and $y(t)$ are the input and the output of the system, respectively.

(a) (5 pts) Find the impulse response function $h(t)$.

(b) (2 pts) Is the system C or NC? Provide justification.

(c) (5 pts) Is this system BIBO stable? Provide justification.

Solution:

a)

We can find $h(t)$ directly from the convolution integral. We first simplify the expression for y(t).

$$\begin{aligned} y(t) &= \int_{-\infty}^{t-1} e^{t-\tau+2} cos(2(t-\tau-1))x(\tau)d\tau \\ &= e^3 \int_{-\infty}^{\infty} e^{t-\tau-1} cos(2(t-\tau-1))u(t-\tau-1)x(\tau)d\tau \end{aligned}$$

We see that $h(t-\tau, 0) = e^3 e^{t-\tau-1} cos(2(t-\tau-1))u(t-\tau-1)$. We can abuse the notation and express the IRF in terms of one variable:

$$h(t) = e^3 e^{t-1} cos(2(t-1))u(t-1).$$

b)

The system is causal since $h(t)$ is a causal function. This could be concluded from the very first expression for $y(t)$ where integration goes up to $t-1$, which means that $y(t)$ depends only on the past.

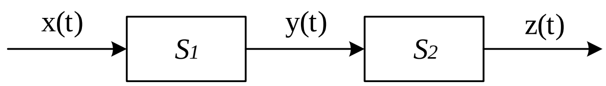
c)

The system is not BIBO stable since $h(t)$ is not absolutely integrable.

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |e^3 e^{t-1} cos(2(t-1))u(t-1)|dt$$
$$= e^3 \int_{1}^{\infty} |e^{t-1}||cos(2(t-1))|dt \to \infty$$

Problem 4 (10 pts)

Consider a cascade of two systems $S_{12} = S_1 S_2$.



The first system S_1 is described by:

$$y(t) = \int_0^t e^{-\sigma} t x(\sigma) d\sigma, \quad t > 0$$

where $x(t)$ and $y(t)$ are the input and the output, respectively. The second system is described by:

$$z(t) = \int_2^t \sigma y(\sigma) d\sigma, \quad t > 2$$

where $y(t)$ and $z(t)$ and the input and the output, respectively.

Find the impulse response function $h_{12}(t, \tau)$ of the cascaded system S_{12}.

Solution:

We first find IRF of S_1 by using $x(\sigma) = \delta(\sigma - \tau)$.

$$\begin{aligned} h_1(t, \tau) &= \int_0^t e^{-\sigma} t \delta(\sigma - \tau) d\sigma \\ &= t \int_0^t e^{-\sigma} \delta(\sigma - \tau) d\sigma \\ &= \begin{cases} te^{-\tau}, & \tau > 0, t - \tau > 0 \\ 0, & \text{otherwise} \end{cases} \\ &= te^{-\tau} u(\tau) u(t - \tau) \end{aligned}$$

We can find $h_{12}(t, \tau)$ if we plug-in $h_1(t, \tau)$ in the expression for $z(t)$. In other words, we can use $h_1(t,\tau)$ as the input to system S_2 to find $h_{12}(t,\tau)$.

$$\begin{aligned}
h_{12}(t,\tau) &= \int_2^t \sigma h_1(\sigma,\tau)d\sigma \\
&= \int_2^t \sigma\sigma e^{-\tau}u(\tau)u(\sigma-\tau)d\sigma \\
&= e^{-\tau}u(\tau)\int_2^t \sigma^2 u(\sigma-\tau)d\sigma \\
&= \begin{cases} 0, & \tau > t \\ e^{-\tau}u(\tau)\int_\tau^t \sigma^2 d\sigma, & 2 < \tau \leq t \\ e^{-\tau}u(\tau)\int_2^t \sigma^2 d\sigma, & \tau \leq 2 \end{cases} \\
&= \begin{cases} 0, & \tau > t \\ \frac{1}{3}e^{-\tau}u(\tau)(t^3-\tau^3), & 2 < \tau \leq t \\ \frac{1}{3}e^{-\tau}u(\tau)(t^3-8), & \tau \leq 2 \end{cases}
\end{aligned}$$

Problem 5 (15 pts)

1) (5 pts) Find the Laplace transform and ROC of the following function.

   $f(t) = te^{-3t}(sin(t) + cos(t))^2 u(t)$.

   Hint: $sin(2x) = 2sin(x)cos(x)$.

2) (10 pts) Find the functions that correspond to the following Laplace transforms. Assume that time-domain functions are causal.

   (a) (5 pts) $F(s) = \frac{s}{(s+1)^2+3}$

   (b) (5 pts) $F(s) = \frac{1}{s^4}(1 - e^{-s})^2$.

Solution:

1)

$$\begin{aligned}\mathcal{L}\{(sin(t) + cos(t))^2 u(t)\} &= \mathcal{L}\{(sin^2(t) + cos^2(t) + 2sin(t)cos(t))u(t)\} \\ &= \mathcal{L}\{(1 + sin(2t))u(t)\} \\ &= \frac{1}{s} + \frac{2}{s^2+4}, \quad \mathcal{R}e\{s\} > 0\end{aligned}$$

$$\begin{aligned}\mathcal{L}\{t(sin(t) + cos(t))^2 u(t)\} &= \frac{-d}{ds}\{\frac{1}{s} + \frac{2}{s^2+4}\} \\ &= \frac{1}{s^2} + \frac{4s}{(s^2+4)^2}, \quad \mathcal{R}e\{s\} > 0\end{aligned}$$

$$\mathcal{L}\{te^{-3t}(sin(t) + cos(t))^2 u(t)\} = \frac{1}{(s+3)^2} + \frac{4(s+3)}{((s+3)^2+4)^2}, \quad \mathcal{R}e\{s\} > -3$$

2)

a)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+3}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+3}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3}+\frac{-1}{(s+1)^2+3}\right\} \\
&= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}+\frac{-1}{s^2+3}\right\} \\
&= e^{-t}(cos(\sqrt{3}t)-\frac{sin(\sqrt{3}t)}{\sqrt{3}})u(t)
\end{aligned}$$

b)

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{6}u(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}(1-2e^{-s}+e^{-2s})\right\} = \frac{1}{6}(t^3u(t)-2(t-1)^3u(t-1)+(t-2)^3u(t-2))$$

Problem 6 (15 pts)

Consider an LTI system S with the following impulse response function

$$h(t) = (\frac{2}{3}e^{-t+1} + \frac{1}{3}e^{2t-2})u(t-1)$$

(a) (7 pts) Find the transfer function $H(s)$. Sketch its zero-pole plot and then denote ROC in the plot.

(b) (8 pts) If the output of the system is

$$y(t) = (-\frac{1}{5}cos(t-2) - \frac{2}{5}sin(t-2) + \frac{1}{5}e^{2(t-2)})u(t-2)$$

find its corresponding causal input $x(t)$.

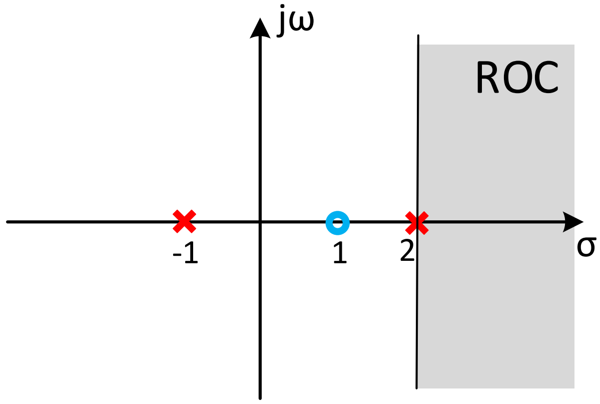
Solution:

a)

The transfer function $H(s)$ is

$$\begin{aligned} H(s) &= (\frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2})e^{-s} \\ &= \frac{2(s-2)+s+1}{3(s+1)(s-2)}e^{-s} \\ &= \frac{s-1}{(s+1)(s-2)}e^{-s} \end{aligned}$$

There is one zero at $s=0$, and two poles at $s=-1$ and $s=2$. The region of convergence is $\mathcal{R}e\{s\} > 2$, as depicted in the figure.



b)

The Laplace transform of the output is

$$\begin{aligned}Y(s) &= \left(-\frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} + \frac{1}{5}\frac{1}{s-2}\right)e^{-2s} \\ &= \frac{-s(s-2) - 2(s-2) + s^2 + 1}{5(s^2+1)(s-2)}e^{-2s} \\ &= \frac{1}{(s^2+1)(s-2)}e^{-2s}\end{aligned}$$

We can find the Laplace transform of $x(t)$ from the fact that $Y(s) = H(s)X(s)$.

$$\begin{aligned}X(s) = \frac{Y(s)}{H(s)} &= \frac{\frac{1}{(s^2+1)(s-2)}e^{-2s}}{\frac{s-1}{(s+1)(s-2)}e^{-s}} \\ &= \frac{s+1}{(s^2+1)(s-1)}e^{-s}\end{aligned}$$

We can find $x(t)$ by calculating the inverse Laplace transform of $X(s)$. First, we represent $X(s)$ using partial fractions.

$$X(s) = \frac{s+1}{(s^2+1)(s-1)}e^{-s} = \left(\frac{As+B}{s^2+1} + \frac{C}{s-1}\right)e^{-s}$$

$$s + 1 = (A + C)s^2 + (-A + B)s - B + C$$

We find that $A = -1$, $B = 0$, $C = 1$. The Laplace transform $X(s)$ can be expressed as

$$X(s) = \left(-\frac{s}{s^2 + 1} + \frac{1}{s - 1} \right) e^{-s}$$

Finally, $x(t)$ is

$$x(t) = (-cos(t - 1) + e^{t-1})u(t - 1)$$