UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm I Solutions
Winter Quarter 2017

Problem 1 (20 pts)

(a) The signals after basic operations are

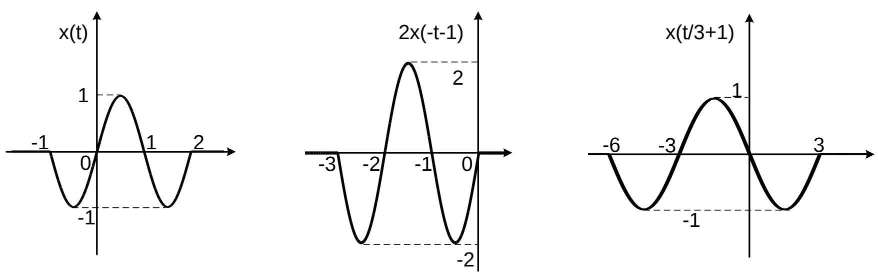


Figure 1: Problem 1 (a)

(b) Energy of $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^{2} \sin^2(\pi t) dt = \int_{-1}^{2} [0.5 - 0.5\cos(2\pi t)]\, dt$$
$$= 1.5 - 0.5 \int_{-1}^{2} \cos(2\pi t) dt = 1.5 - 0.25 \sin(2\pi t)|_{-1}^{2} = 1.5$$

Energy of $2x(-t-1), x(t/3+1)$ are 6 and 4.5, respectively. You can use similar integral or resort to general results from part (c).

(c) The energy is

$$\int_{-\infty}^{\infty} [Ax(Bt+C)]^2\, dt = A^2 \int_{-\infty}^{\infty} x^2(Bt+C) dt$$

Use change of variable $\sigma = Bt + C$, the above equation can be written as

$$\frac{A^2}{B}\int_{-\infty}^{\infty} x^2(\sigma)d\sigma = \frac{A^2}{B}E_x, \text{ if } B > 0$$
$$\frac{-A^2}{B}\int_{-\infty}^{\infty} x^2(\sigma)d\sigma = \frac{-A^2}{B}E_x, \text{ if } B < 0$$

Therefore energy of new signal is $\frac{A^2 E_x}{|B|}$ where $E_x = 3/2$ is the energy of original signal $x(t)$.

(d) The even and odd components are shown in Figure 2.

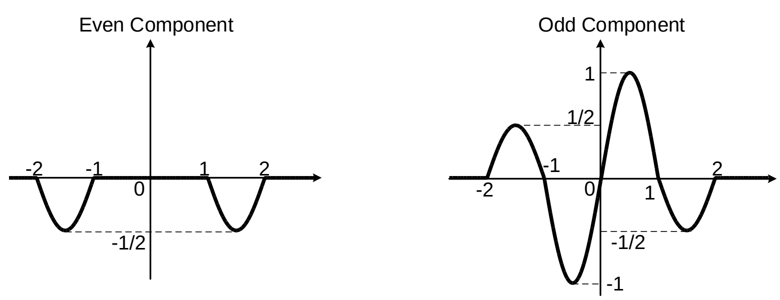


Figure 2: Problem 1 (d)

Problem 2 (20 pts)

(a) The system is time-invariant since IRF $h(t, \tau)$ is function of $t - \tau$. We can further write is as $h(t) = e^{-2t}u(t)$

(b) The system is causal since IRF $h(t, \tau)$ is zero when $t < \tau$.

(c) The system is BIBO stable since

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{0}^{\infty} e^{-2t}dt = 0.5$$

is a finite value.

(d) The IPOP relation can be rewritten as

$$y(t) = \int_{-\infty}^{\infty} e^{-2(t-\tau)}u(t-\tau)x(\tau)d\tau.$$

The output associated with $x_1(t) = \delta(t-1)$ is

$$\begin{aligned} y_1(t) = &\int_{-\infty}^{\infty} e^{-2(t-\tau)}u(t-\tau)\delta(\tau-1)d\tau \\ =&e^{-2(t-1)}u(t-1)\int_{-\infty}^{\infty} \delta(\tau-1)d\tau \\ =&e^{-2(t-1)}u(t-1) \end{aligned}$$

(e) The output associated with $x_2(t) = u(1-t)$ is

$$y_2(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)u(1-\tau)d\tau$$

If $t > 1$

$$y_2(t) = \int_{-\infty}^{1} e^{-2(t-\tau)}d\tau = e^{-2t}\int_{-\infty}^{1} e^{2\tau}d\tau = 0.5e^{-2t+2}$$

If $t \leq 1$

$$y_2(t) = \int_{-\infty}^{t} e^{-2(t-\tau)}d\tau = e^{-2t}\int_{-\infty}^{t} e^{2\tau}d\tau = 0.5.$$

Problem 3 (15 pts)
The IPOP can be written as:

$$y(t) = \int_{-\infty}^{\infty} e^{t-\tau}\sin[2(t-\tau)-4]u(t-\tau)x(\tau)d\tau \tag{1}$$

Therefore,

(a) IRF is $h(t,\tau) = e^{t-\tau}\sin[2(t-\tau)-4]u(t-\tau) = h(t-\tau)$ and $h(t) = e^t \sin(2t-4)u(t)$.

(b) The system is C, because $h(t,\tau) = 0$ for $t < \tau$ or $h(t) = 0$ for $t < 0$. Alternatively, the system is C because $y(t)$ depends on inputs upto time t.

(c) The system is TI, since $h(t,\tau) = h(t-\tau)$.

(d) We have $\int_{-\infty}^{\infty} |h(t)|dt = \int_0^\infty e^t |\sin(2t-4)|dt \to \infty$ because $e^t \to \infty$ at $t \to \infty$. Therefore, the system is not BIBO stable.

Problem 4 (20 pts)

(a) The signal $x_2(t)$ can be written as

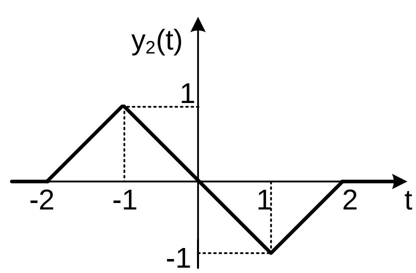
$$x_2(t) = x_1(t+1) - x_1(t)$$

Therefore, $a_1 = 1, \tau_1 = -1, a_2 = -1, \tau_2 = 0$.

(b) Using the property of LTI system, the output signal corresponding to $x_2(t)$ can be written as

$$y_2(t) = y_1(t+1) - y_1(t)$$

therefore the signal $y_2(t)$ has following shape



Problem 5 (20 pts)

(a) $h_2(t) = u(\alpha - t)u(t)$ or $h_2(t) = u(t) - u(t-a)$.

(b) Applying the impulse at input of S_1, $x(t) = \delta(t-\tau)$, we get the IRF of S_1:
$h_1(t,\tau) = \delta(t-\tau)\cos(2\pi f_0 t)$. Then, applying the IRF $h_1(t,\tau)$ at input of S_2, we get IRF h_{12}:

$$h_{12}(t,\tau) = \int_{-\infty}^{\infty} h_2(t,\sigma)h_1(\sigma,\tau)d\sigma \tag{2}$$

(i) Method 1: Using $h_2(t) = u(\alpha - t)u(t)$
We have $h_2(t,\sigma) = h_2(t-\sigma) = u(\alpha - t + \sigma)u(t-\sigma)$ and $h_1(\sigma,\tau) = \delta(\sigma-\tau)\cos(2\pi f_0\sigma)$. Therefore,

$$h_{12}(t,\tau) = \int_{-\infty}^{\infty} \delta(\sigma-\tau)\cos(2\pi f_0\sigma)u(\alpha - t + \sigma)u(t-\sigma)d\sigma \tag{3}$$
$$= \cos(2\pi f_0\tau)u(\alpha - t + \tau)u(t-\tau). \tag{4}$$

The last equality is obtained by substituting $\sigma = \tau$ in remaining integrand by using property of the impulse.

(ii) Method 2: Using $h_2(t) = u(t) - u(t-\alpha)$
We have $h_2(t,\sigma) = h_2(t-\sigma) = u(t-\sigma) - u(t-\sigma-\alpha)$ and $h_1(\sigma,\tau) = \delta(\sigma-\tau)\cos(2\pi f_0\sigma)$. Therefore,

$$h_{12}(t,\tau) = \int_{-\infty}^{\infty} \delta(\sigma-\tau)\cos(2\pi f_0\sigma)\left[u(t-\sigma) - u(t-\sigma-\alpha)\right]d\sigma \tag{5}$$
$$= \cos(2\pi f_0\tau)\left[u(t-\tau) - u(t-\tau-\alpha)\right]. \tag{6}$$

The last equality is obtained by substituting $\sigma = \tau$ in remaining integrand by using property of the impulse.

(c) S_1S_2 in TV, because $h_{12}(t,\tau) \neq h_{12}(t-\tau)$.

(d) The system is C, because $h_{12}(t,\tau) = 0$ for $t < \tau$.

Problem 6 (15 pts)

(a) System is C, because $h(t) = 0$ for $t < 0$.

(b)

$$
\begin{aligned}
H(s) &= \mathcal{L}[\cos(2\pi t)u(t)] + \mathcal{L}[\sin(4\pi t)u(t)] \\
&= \frac{s}{s^2+4\pi^2} + \frac{4\pi}{s^2+16\pi^2}, \text{ROC:}\Re(s) > 0 \qquad (7)
\end{aligned}
$$

(c) Using eigen-function property of exponential functions:

$$
\begin{aligned}
y(t) &= e^{2t} H(s=2) \\
&= e^{2t} \times \left(\frac{2}{4+4\pi^2} + \frac{4\pi}{4+16\pi^2} \right) \\
&= e^{2t} \times \left(\frac{1}{2+2\pi^2} + \frac{\pi}{1+4\pi^2} \right) \\
&= e^{2t} \times \left(\frac{1+2\pi+4\pi^2+2\pi^3}{2+10\pi^2+8\pi^4} \right), t \in (-\infty, \infty). \qquad (8)
\end{aligned}
$$

Additional problem 1

(a) False. The result of $y(t) = \cos(2\pi t) * h(t)$ can be interpreted as output of LTI system whose IRF is $h(t)$ and the input is $\cos(2\pi t)$. Due to the eigenfunction property, the output is

$$
y(t) = \frac{1}{2} H(2\pi j) e^{2j\pi t} + \frac{1}{2} H(-2\pi j) e^{-2j\pi t}
$$

Therefore it is always $A\cos(2\pi t - \theta)$.
Grading comments: Full credit is not given if one simply states "frequency will not change" without further reasoning.

(b) False. Two poles are at $s = 1 \pm j$, which are in right half plane. Therefore the ROC does not contains $j\Omega$ axis.

Additional problem 2

(a)

$$\begin{aligned} x(t) &= \cos(3t)u(t-2\pi) \\ &= \cos(3(t-2\pi)+6\pi)u(t-2\pi) \\ &= \cos(3(t-2\pi))u(t-2\pi) \end{aligned}$$

Using time shift property of Laplace transform:

$$\begin{aligned} X(s) &= e^{-2\pi s}\mathcal{L}\left[\cos(3t)u(t)\right] \\ &= \frac{se^{-2\pi s}}{s^2+9} \end{aligned}$$

ROC is $\mathcal{R}e[s] > 0$.
Poles are at $s = \pm 3j$ and zero is at $s = 0$.

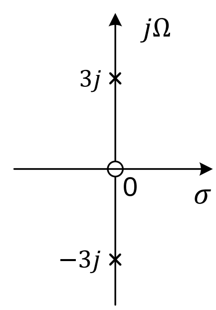


Figure 3: Pole-zero plot for Problem 2-a

(b)

$$\begin{aligned} y(t) &= \int_0^t (t-\tau)^3\cos(3\tau)d\tau \\ &= \int_{-\infty}^{\infty} (t-\tau)^3\cos(3\tau)u(\tau)u(t-\tau)d\tau \\ &= [t^3u(t)] * [\cos(3t)u(t)] \end{aligned}$$

Using convolution property of Laplace transform:

$$Y(s) = \mathcal{L}[t^3 u(t)] \times \mathcal{L}[\cos(3t)u(t)]$$

Consider term I

$$\mathcal{L}[t^3 u(t)] = 3!\mathcal{L}\left[\frac{t^3}{3!}u(t)\right] = 3!\frac{1}{s^4} = \frac{6}{s^4}$$

It has ROC: $\mathcal{R}e[s] > 0$.

Consider term II

$$\mathcal{L}[\cos(3t)u(t)] = \frac{s}{s^2+9}$$

It has ROC: $\mathcal{R}e[s] > 0$.

Therefore,

$$Y(s) = \frac{6}{s^4} \times \frac{s}{s^2+9} = \frac{6}{s^3(s^2+9)}$$

and ROC: $\mathcal{R}e[s] > 0$.
There are 5 poles in total. Three poles are at $s = 0$ and two poles are at $s = \pm 3j$. There are no zeros.

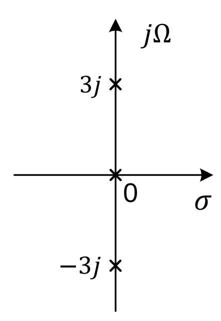


Figure 4: Pole-zero plot for Problem 2-b