EE 102 (Spring 2014)

Mid term Solution

Q.1 (15)

① IPOP: $y(t) = \int_{-\infty}^{\infty} t(t-\tau)\, u(t-\tau)\, x(\tau)\, d\tau, \quad t \in (-\infty, \infty)$

(10)

$$\boxed{\text{IRF } h(t,\tau) = t(t-\tau)\, u(t-\tau)}$$

using $x(\tau) = u(\tau)\, u(3-\tau)$

$$y(t) = \int_{-\infty}^{\infty} t(t-\tau)\, u(t-\tau)\, u(\tau)\, u(3-\tau)\, d\tau$$

$y(t) = 0$ for $t < 0$

$$y(t) = \int_{0}^{t} t(t-\tau)\, d\tau, \quad 0 \le t < 3$$

$$= t \cdot \left[t^2 - \frac{t^2}{2}\right]$$

$$= \frac{t^3}{2}.$$

$$y(t) = \int_{0}^{3} t(t-\tau)\, d\tau, \quad t \geq 3$$

$$= t\left[t \cdot \tau - \frac{\tau^2}{2}\right]_0^3 = t\left[3t - \frac{9}{2}\right]$$

$$\therefore\ y(t) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{t^3}{2} & , \quad 0 \le t < 3 \\ t\left[3t - \dfrac{9}{2}\right] & ; \quad t \geq 3 \end{cases}$$

(ii) From IRF : $h(t,\tau) = t\,(t-\tau)u(t-\tau)$

5

S is TV and Causal.

$h(t,\tau) = t(t-\tau)u(t-\tau)$

Q.2

15

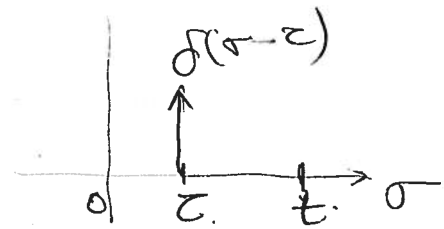
$$\cancel{h_1(t)} \quad h_1(t,\tau) = e^{-t}\,\delta(t-\tau)\,u(t) = \cancel{\#}$$

$$\cancel{h_2(t,\tau)} =$$

$$h_{12}(t,\tau) = \int_0^t e^{-(t-\sigma)}\, h_1(\sigma,\tau)\, u(\sigma)\, d\sigma \quad , t \geq 0$$

$$= \int_0^t e^{-(t-\sigma)}\, e^{-\sigma}\, \delta(\sigma-\tau)\, u(\sigma)\, d\sigma$$

$$= e^{-t} \int_0^t \delta(\sigma-\tau)\, d\sigma.$$



$$= e^{-t}\, u(t-\tau)\, u(\tau)$$

Q.3 15

(i) 5

$$H(s) = \frac{K.(s-2)}{(s-j)(s+j)(s+1)} \qquad k = \text{const.}$$

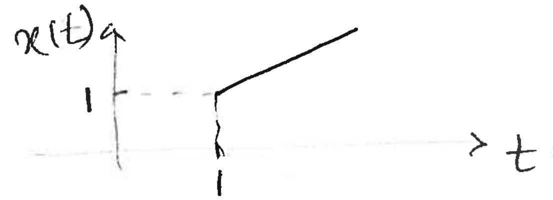
$$= \frac{K.(s-2)}{(s^2+1)(s+1)}.$$

$$H(0) = \frac{K.(-2)}{1} = \sqrt{2}. \Rightarrow K = \frac{-1}{\sqrt{2}}.$$

$$\therefore \boxed{H(s) = \frac{-1}{\sqrt{2}} \frac{(s-2)}{(s^2+1)(s+1)}}.$$

(ii) 10

$$x(t) = t\, u(t-1)$$



$$= (t-1)\, u(t-1) + u(t-1)$$

$$\therefore X(s) = e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]$$

$$= \frac{e^{-s}}{s^2}\left[s+1\right]$$

$$Y(s) = H(s) \cdot X(s)$$

$$= \frac{-1}{\sqrt{2}} \frac{(s-2)}{(s^2+1)\cancel{(s+1)}} \frac{e^{-s}\cancel{(s+1)}}{s^2}.$$

$$= \frac{-e^{-s}}{\sqrt{2}}\left(\frac{s-2}{(s^2+1)\,s^2}\right] \quad \text{—— (1)}$$

Let $Y_1(s) = \dfrac{s-2}{s^2(s^2+1)} = \dfrac{As+B}{s^2} + \dfrac{Cs+D}{s^2+1}$

* Comparing coefficients of equal powers of s on both sides.

for s^3: $A+C=0$

s^2: $B+D=0$

s: $A=1$

const: $B=-2$.

$\Rightarrow D=2,\ C=-1$

$$\therefore Y_1(s) = \frac{s-2}{s^2} - \frac{s-2}{s^2+1}$$

$$= \frac{1}{s} - \frac{2}{s^2} - \frac{s}{s^2+1} + \frac{2}{s^2+1}$$

$$y_1(t) = u(t) - 2t\,u(t) - \cos(t)u(t) + 2\sin(t)u(t)$$

From ①. $Y(s) = -\dfrac{e^{-s}}{\sqrt{2}} Y_1(s)$.

$$\therefore y(t) = \frac{-1}{\sqrt{2}} y_1(t-1)$$

$$= \frac{-1}{\sqrt{2}}\Big[u(t-1) - 2(t-1)u(t-1) - \cos(t-1)u(t-1) + 2\sin(t-1)u(t-1)\Big]$$

$3-2t$

$-2t+1 \cancel{\rightarrow 3-2t)u(t-1)}$

Q.4

$$X(s) = \frac{\sqrt{2}}{s+1}$$

$$Y(s) = \frac{\sqrt{2}}{s+1} - \frac{2\sqrt{2}}{(s+1)^2}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{\sqrt{2}(s+1)-2\sqrt{2}}{(s+1)^2}}{\frac{\sqrt{2}}{s+1}} = \frac{(s+1)-2}{s+1} = \frac{s-1}{s+1}$$

$$= \frac{s}{s+1} - \frac{1}{s+1} = 1 - \frac{2}{s+1}$$

$$\Rightarrow h(t) = \cancel{\cos t\, u(t) - \sin t\, u(t)}$$

$$= \delta(t) - 2e^{-t}u(t)$$ #

Q.5.

(i) $$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+s+1}$$

$$\Rightarrow s^2Y(s) + sY(s) + Y(s) = X(s)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = x(t)$$

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(ii) $\quad x(t) = \sin(2(t-1))\,u(t-1)$

$$\Rightarrow X(s) = \frac{2e^{-s}}{s^2+4}$$

$$\Rightarrow Y(s) = H(s)X(s) = \frac{1}{s^2+s+1} \cdot \frac{2e^{-s}}{s^2+4}$$

$$= \left[\frac{2}{(s^2+4)(s^2+s+1)}\right] e^{-s}$$

$$= G(s)\, e^{-s}$$

$$G(s) = \frac{2}{(s^2+4)(s^2+s+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+s+1}$$

$$\Rightarrow \quad A = \frac{-2}{13},\ B = \frac{-6}{13},\ C = \frac{2}{13},\ D = \frac{8}{13}.$$

$$\Rightarrow G(s) = A\frac{s}{s^2+4} + B\cdot\frac{1}{s^2+4} + \frac{C(s+\frac{1}{2}) + D}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow g(t) = A\cos(2t)u(t) + \frac{B}{2}\sin(2t)u(t) + Ce^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t) + \frac{2D}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

$$\Rightarrow \quad y(t) = g(t-1)$$ ※

(iii) ① Direct Calculation.

$$H(s) = \frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \Rightarrow h(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) u(t)$$

$$\Rightarrow \quad g(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-\tau)} \sin\left(\frac{\sqrt{3}}{2}(t-\tau)\right) u(t-\tau) u(\tau) d\tau$$

$$= \int_0^t \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} e^{\frac{1}{2}\tau} \sin\left(\frac{\sqrt{3}}{2}(t-\tau)\right) d\tau$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \int_0^t e^{\frac{1}{2}\tau} \left[\frac{e^{j[\frac{\sqrt{3}}{2}(t-\tau)]} - e^{-j[\frac{\sqrt{3}}{2}(t-\tau)]}}{2j}\right] d\tau$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \int_0^t \frac{1}{2j} \left[e^{j\frac{\sqrt{3}}{2}t} \cdot e^{(\frac{1}{2}-\frac{\sqrt{3}}{2}j)\tau} - e^{-j\frac{\sqrt{3}}{2}t} \cdot e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}j)\tau}\right] d\tau$$

$$= \frac{\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t}}{2j} e^{j\frac{\sqrt{3}}{2}t} \left[\frac{e^{(\frac{1}{2}-\frac{\sqrt{3}}{2}j)t} - 1}{\frac{1}{2}-\frac{\sqrt{3}}{2}j}\right] - \frac{\frac{2}{\sqrt{3}} e^{-\frac{1}{2}t}}{2j} e^{-j\frac{\sqrt{3}}{2}t} \left[\frac{e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}j)t} - 1}{\frac{1}{2}+\frac{\sqrt{3}}{2}j}\right]$$

$$= 1 - e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) - \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) \ \#$$

② By Laplace

$$X(s) = \frac{1}{s} \Rightarrow G(s) = H(s) X(s) = \frac{1}{s(s^2+s+1)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$\Rightarrow A = 1,\ B = -1,\ C = -1.$$

$$\Rightarrow G(s) = \frac{1}{s} + \frac{-s-1}{s^2+s+1} = \frac{1}{s} - \frac{s+1}{s^2+s+1}$$

$$G(s) = \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right) + \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow \quad g(t) = u(t) - e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

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Q.6.

$$y(t) = \int_{-\infty}^{\infty} h(t-\sigma)\,u(t-\sigma)\,x(\sigma)\,d\sigma$$

$$= h(t)u(t) * x(t)$$

(i)

$$h(t) = t^2 u(t), \quad x(t) = e^{-t}u(t-5)$$

$$\Rightarrow H(s) = -\frac{d\,\frac{1}{s^2}}{ds} = \frac{2}{s^3}, \quad X(s) = e^{-5}\cdot\frac{e^{-5s}}{s+1}$$

$$\Rightarrow \quad Y(s) = H(s)\cdot X(s) = \frac{2}{s^3}\cdot\frac{e^{-5s}\cdot e^{-5}}{s+1}$$

$$= e^{-5s}\cdot 2e^{-5}\left[\frac{1}{(s+1)s^3}\right]$$

$$\frac{1}{(s+1)s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}, \quad \Rightarrow \quad C=1,\ B=-1,\ A=1,\ D=-1$$

$$= \frac{1}{s} + \frac{-1}{s^2} + \frac{1}{s^3} + \frac{-1}{s+1}$$

$$\Rightarrow \quad y(t) = 2e^{-5}\left[u(t-5) - (t-5)u(t-5) + \frac{1}{2}(t-5)^2u(t-5) - e^{-(t-5)}u(t-5)\right]$$

(ii) $h(t) = \sin(wt)$ $\quad$ $x(t) = t\,u(t)$

$H(s) = \dfrac{w}{s^2 + w^2}$ $\quad$ $X(s) = \dfrac{1}{s^2}$

$\Rightarrow Y(s) = \dfrac{w}{s^2(s^2+w^2)} = \dfrac{As+B}{s^2} + \dfrac{Cs+D}{s^2+w^2}$

$\Rightarrow A = 0, \; B = \dfrac{1}{w}, \; C = 0, \; D = \dfrac{-1}{w}$

$\Rightarrow Y(s) = \dfrac{1}{w} \cdot \dfrac{1}{s^2} + \dfrac{\frac{-1}{w}}{s^2+w^2}$

$\Rightarrow y(t) = \dfrac{1}{w} t\,u(t) - \dfrac{1}{w^2} \sin(wt)\,u(t)$

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