# EE 102: Midterm Solution – Spring 2012

Problem 1:

$$y(t) = \int_{-\infty}^{\infty} t(t-\tau)u(t-\tau)\, x(\tau)d\tau$$

i) Written as a convolution integral, therefore $h(t,\tau) = t(t-\tau)u(t-\tau)$

When $x(t) = U(t)U(3-t)$, we have

$$t < 0 : y(t) = \int_{-\infty}^{t} t(t-\tau) \cdot 1 \cdot 0 d\tau = 0$$

$$0 \le t \le 3: y(t) = \int_{-\infty}^{0} t(t-\tau) \cdot 1 \cdot 0\, d\tau + \int_{0}^{t} t(t-\tau) \cdot 1 \cdot 1\, d\tau = 0 + \frac{t^3}{2}$$

$$t > 3 : y(t) = 0 + \int_{3}^{t} t(t-\tau) \cdot 1 \cdot 1\, d\tau + 0 = 3t^2 + \frac{9t}{2}$$

ii) TV, C

Problem 2:

i) $S_1$ written as a convolution integral

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}U(t-\tau)x(\tau)d\tau$$

Therefore $h_1(t) = e^{-t}U(t)$.

$S_2$ written as a convolution integral

$$w(t) = \int_{-\infty}^{\infty} [\delta(t-\tau) - U(t-\tau)]v(\tau)d\tau$$

Therefore $h_2(t) = \delta(t) - U(t)$.

$$\begin{aligned} h_{21}(t) &= \int_{-\infty}^{\infty} h_1(t-\tau)h_2(\tau)d\tau \\ &= \int_{0}^{t} e^{-(t-\tau)}[\delta(\tau) - U(\tau)]d\tau \\ &= (2e^{-t} - 1)U(t) \end{aligned}$$

ii) Since both systems can be expressed in the convolutional integral form with impulse responses of the form $h(t-\tau)$ then both systems are LTI. Also, since their $h(t) > 0$ only for $t \ge 0$ then the systems are causal. So both systems are LTIC.

Problem 3:

i) $H(s) = \frac{L_s\{e^{-t}t - e^{-t}t^2\}}{L_s\{e^{-t}t\}} = \frac{\frac{1}{(s+1)^2} - \frac{2}{(s+1)^3}}{\frac{1}{(s+1)^2}} = \frac{s-1}{s+1} = 1 - \frac{2}{s+1}$

$$h(t) = \delta(t) - 2e^{-t}U(t)$$

ii) $\frac{Y(s)}{X(s)} = \frac{L_s\{\cos(t)U(t)\}}{X(s)} = H(s) = \frac{s-1}{s+1} \rightarrow X(s) = \frac{s}{s^2+1} \cdot \frac{s+1}{s-1}$

$$\frac{s^2+s}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Cs+D}{s^2+1} = \frac{1}{s-1} + \frac{Cs+D}{s^2+1} = \frac{(1+C)s^2 + (D-C)s - D + 1}{(s-1)(s^2+1)}$$

$D = 1, C = 0$

$X(s) = \frac{1}{s-1} + \frac{1}{s^2+1} \rightarrow x(t) = (e^t + \sin(t))U(t)$

Problem 4:

$$L_s\{e^{-t}\{\int_0^t \cos(t-\tau)\, e^{-\tau}\sin(\tau)\, d\tau\}\} = L_s\{e^{-t}g(t)\}$$

$$L_s\{g(t)\} = L_s\{cost(t)U(t)\} \cdot L_s\{e^{-\tau}\sin(\tau)\, U(\tau)\} = \frac{s}{s^2+1} \cdot \frac{1}{(s+1)^2+1}$$

$$L_s\{e^{-t}g(t)\} = \frac{s+1}{(s+1)^2+1} \cdot \frac{1}{(s+2)^2+1}$$

Problem 5:

There is no unique solution to this problem. The problem could be solved using Laplace transforms by writing $F(s) = X(s)H(s)$ for any $X(s)$ and $H(s)$ of your choice, which results in

$$f(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

Another simple way to solve this problem is to convolve the given $f(t)$ with an impulse function by taking advantage of the sifting property,

$$f(t) = \int_{-\infty}^{\infty} (\cos(\tau) + \sin(\tau) - 1)U(\tau)\delta(t-\tau)d\tau$$

Problem 6:

Impulse Response:

$$L_s\left\{\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t)\right\}$$

$$Y(s)(s^2+s+1) = X(s)(s+1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+s+1}$$

$$H(S) = \frac{\left(s+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{\sqrt{3}}\right)}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$h(t) = e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]U(t)$$

Unit step response:

$$G(s) = \frac{H(s)}{s} = \frac{s+1}{s(s^2+s+1)}$$

$$G(s) = \frac{1}{s} - \frac{s}{s^2+s+1}$$

$$g(t) = U(t) - e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]U(t)$$

<u>Problem 7:</u>

In convolutional form

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}U(t-\tau)x(\tau)d\tau + \int_{-\infty}^{\infty} U(t-\tau)x(\tau)d\tau$$

$$h(t) = (e^{-t}+1)U(t)$$

$$H(s) = \frac{1}{s+1} + \frac{1}{s} = \frac{2s+1}{s^2+s} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s^2+s) = X(s)(2s+1)$$

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} = 2\frac{dx(t)}{dt} + x(t)$$