UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm II Solutions
Winter Quarter 2017

Problem 1 (8 pts)

(a) False. The result of $y(t) = \cos(2\pi t) * h(t)$ can be interpreted as output of LTI system whose IRF is $h(t)$ and the input is $\cos(2\pi t)$. Due to the eigenfunction property, the output is

$$y(t) = H(2\pi j)e^{2j\pi t} + H(-2\pi j)e^{-2j\pi t}$$

Therefore it is always $A\cos(2\pi t - \theta)$.
Grading comments: Full credit is not given if one simply states "frequency will not change" without further reasoning.

(b) False. Two poles are at $s = 1 \pm j$, which are in right half plane. Therefore the ROC does not contains $j\Omega$ axis.

Problem 2 (12 pts)

(a)

$$\begin{aligned} x(t) &= \cos(3t)u(t-2\pi) \\ &= \cos(3(t-2\pi)+6\pi)u(t-2\pi) \\ &= \cos(3(t-2\pi))u(t-2\pi) \end{aligned}$$

Using time shift property of Laplace transform:

$$\begin{aligned} X(s) &= e^{-2\pi s}\mathcal{L}\left[\cos(3t)u(t)\right] \\ &= \frac{se^{-2\pi s}}{s^2+9} \end{aligned}$$

ROC is $\mathcal{R}e[s] > 0$.
Poles are at $s = \pm 3j$ and zero is at $s = 0$.

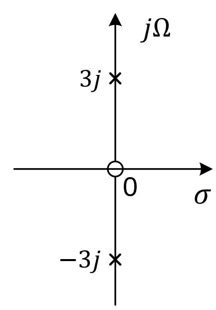

Figure 1: Pole-zero plot for Problem 2-a

(b)

$$\begin{aligned} y(t) &= \int_0^t (t-\tau)^3 \cos(3\tau)d\tau \\ &= \int_{-\infty}^{\infty} (t-\tau)^3 \cos(3\tau)u(\tau)u(t-\tau)d\tau \\ &= [t^3u(t)] * [\cos(3t)u(t)] \end{aligned}$$

Using convolution property of Laplace transform:

$$Y(s) = \mathcal{L}[t^3 u(t)] \times \mathcal{L}[\cos(3t)u(t)]$$

Consider term I

$$\mathcal{L}[t^3 u(t)] = 3!\mathcal{L}\left[\frac{t^3}{3!}u(t)\right] = 3!\frac{1}{s^4} = \frac{6}{s^4}$$

It has ROC: $\mathcal{R}e[s] > 0$.

Consider term II

$$\mathcal{L}[\cos(3t)u(t)] = \frac{s}{s^2+9}$$

It has ROC: $\mathcal{R}e[s] > 0$.

Therefore,

$$Y(s) = \frac{6}{s^4} \times \frac{s}{s^2+9} = \frac{6}{s^3(s^2+9)}$$

and ROC: $\mathcal{R}e[s] > 0$.
There are 5 poles in total. Three poles are at $s = 0$ and two poles are at $s = \pm 3j$. There are no zeros.

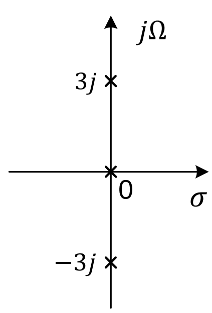

Figure 2: Pole-zero plot for Problem 2-b

Problem 3 (15 pts)

(a) The period is $T_0 = 4$ and fundamental frequency is $\Omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. $X_0 = 0$ since the signal is odd. We use following equation to find remaining coefficients

$$\begin{aligned}
X_k &= \frac{1}{4}\int_{-1}^{1} \sin(\pi t) e^{-jk\Omega_0 t} dt \\
&= \frac{1}{4}\int_{-1}^{1} \frac{e^{j\pi t} - e^{-j\pi t}}{2j} e^{-jk\frac{\pi}{2}t} dt \\
&= \frac{1}{8j}\int_{-1}^{1} (e^{j\pi t(1-k/2)} - e^{-j\pi t(1+k/2)}) dt \\
&= \frac{1}{8j} \times \left[\frac{e^{j\pi t(1-k/2)}|_{-1}^{1}}{j\pi(1-k/2)} - \frac{e^{-j\pi t(1+k/2)}|_{-1}^{1}}{-j\pi(1+k/2)}\right] \\
&= \frac{1}{8j} \times \left[\frac{e^{j\pi(1-k/2)} - e^{-j\pi(1-k/2)}}{j\pi(1-k/2)} - \frac{e^{-j\pi(1+k/2)} - e^{j\pi(1+k/2)}}{-j\pi(1+k/2)}\right] \\
&= \frac{1}{4j} \times \left[\frac{\sin(\pi(1-k/2))}{\pi(1-k/2)} - \frac{\sin(\pi(1+k/2))}{\pi(1+k/2)}\right] \\
&= \frac{j}{4} \times \left[\frac{\sin(\pi(1+k/2))}{\pi(1+k/2)} - \frac{\sin(\pi(1-k/2))}{\pi(1-k/2)}\right]
\end{aligned}$$

(b) Substituting $k = 1$ and $k = -1$, we get

$$\begin{aligned}
X_1 &= \frac{j}{4} \times \left[\frac{\sin(3\pi/2)}{3\pi/2} - \frac{\sin(\pi/2)}{\pi/2}\right] = \frac{j}{4}\left(\frac{-2}{3\pi} - \frac{2}{\pi}\right) = \frac{-2j}{3\pi} \\
X_{-1} &= \frac{j}{4} \times \left[\frac{\sin(\pi/2)}{\pi/2} - \frac{\sin(3\pi/2)}{3\pi/2}\right] = \frac{j}{4}\left(\frac{2}{\pi} + \frac{2}{3\pi}\right) = \frac{2j}{3\pi}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|X_1| = |X_{-1}| &= \frac{2}{3\pi} \\
\angle X_1 = \frac{-\pi}{2}, \angle X_{-1} &= \frac{\pi}{2}
\end{aligned}$$

(c) We have

$$
\begin{aligned}
a_k &= \mathcal{R}e\{X_k\} = 0, \\
b_k &= \mathcal{I}m\{X_k\} = \frac{1}{4} \times \left[\frac{\sin(\pi(1+k/2))}{\pi(1+k/2)} - \frac{\sin(\pi(1-k/2))}{\pi(1-k/2)} \right] \\
x(t) &= -2 \sum_{k=1}^{\infty} \frac{1}{4} \times \left[\frac{\sin(\pi(1+k/2))}{\pi(1+k/2)} - \frac{\sin(\pi(1-k/2))}{\pi(1-k/2)} \right] \sin\left(k\frac{\pi}{2}t\right)
\end{aligned}
$$

Problem 4 (15 pts)

The fundamental frequency is $\Omega_0 = \frac{2\pi}{T_0} = \pi$. Due to the fact $X_k = 0, |k| \geq 3$, we can determine $x(t)$ by

$$x(t) = X_0 + X_1 e^{j\pi t} + X_{-1} e^{-j\pi t} + X_2 e^{j2\pi t} + X_{-2} e^{-j2\pi t}$$

For the DC component, we use

$$X_0 = \frac{1}{2} \int_{-1}^{1} x(t) dt = 1$$

where the second equal sign is because $x(t)$ is an even signal. Due to the fact $x(t)$ is real and even, the Fourier coefficients are real. Let $X_1 = X_{-1} = a$ and $X_2 = X_{-2} = b$. Using $x(0.5) = 3$, we have

$$x(0.5) = 1 + ae^{j\pi/2} + ae^{-j\pi/2} + be^{j\pi} + be^{-j\pi} = 1 + 2b\cos(\pi) = 1 - 2b = 3$$

It gives coefficients $X_2 = X_{-2} = b = -1$.

Due to the power is 3, the Parseval's relation gives

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = |b|^2 + |a|^2 + 1^2 + |a|^2 + |b|^2 = 3 + |a|^2 = 3$$

Therefore $X_1 = X_{-1} = a = 0$ and the time domain signal $x(t)$ is

$$x(t) = 1 - e^{j2\pi t} - e^{-j2\pi t} = 1 - 2\cos(2\pi t)$$

Grading comments: Recall that for integer k, we have

$$\begin{aligned} e^{j\pi k} &= (-1)^k \\ e^{j2\pi k} &= 1. \end{aligned}$$

A common mistake here is to assume equations

$$\begin{aligned} e^{j\pi t} &= (-1)^t \\ e^{j2\pi t} &= 1 \end{aligned}$$

are also valid for $t \in \mathbb{R}$.

Problem 5 (20 pts)

(a) Apply Laplace transform on both side of the equation

$$s^2Y(s) + 2sY(s) + Y(s) = s^2X(s) + X(s)$$

The transfer function of S_1 is

$$\begin{aligned} H_1(s) =& \frac{Y(s)}{X(s)} \\ =& \frac{s^2+1}{s^2+2s+1} \\ =& 1 - \frac{2}{s+1} + \frac{2}{(s+1)^2} \end{aligned}$$

The IRF of S_1 is

$$h_1(t) = \delta(t) - 2e^{-t}u(t) + 2te^{-t}u(t)$$

The IRF of S_2 is

$$h_2(t) = \cos(t)u(t) + \sin(t)u(t)$$

(b) The transfer function of S_2 is

$$H_2(s) = \mathcal{L}\{h_2(t)\} = \frac{s+1}{s^2+1}$$

The cascaded system has transfer function

$$\begin{aligned} H_{12}(s) =& H_1(s) \cdot H_2(s) \\ =& \frac{s^2+1}{s^2+2s+1} \cdot \frac{s+1}{s^2+1} \\ =& \frac{s^2+1}{(s+1)^2} \cdot \frac{s+1}{s^2+1} \\ =& \frac{1}{s+1} \end{aligned}$$

(c) We rewrite input signal using Euler's identity as

$$
\begin{aligned}
x(t) =& 1 + \frac{1}{j}e^{j\pi t} + \frac{-1}{j}e^{-j\pi t} + e^{j3\pi t} + e^{-j3\pi t} \\
=& e^{-j3\pi t} + 0e^{-j2\pi t} + \frac{-1}{j}e^{-j\pi t} + 1e^{j0\pi t} + \frac{1}{j}e^{j\pi t} + 0e^{j\pi 2t} + e^{j3\pi t} + e^{-j3\pi t}
\end{aligned}
$$

The fundamental frequency of $x(t)$ is $\Omega_0 = \pi$ and therefore Fourier series is written as

$$
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\pi t}
$$

Comparing the above two equations, the Fourier coefficient of $x(t)$ can be directly found as

$$
X_k = \begin{cases} 1, & k = 0 \\ \frac{1}{j}, & k = 1 \\ -\frac{1}{j}, & k = -1 \\ 1, & k = 3 \text{ and } k = -3 \\ 0, & \text{otherwise} \end{cases}
$$

Therefore the Fourier coefficient of $z(t)$ are $Z_k = H_{12}(jk\Omega_0)X_k$ and therefore

$$
Z_k = \begin{cases} 1, & k = 0 \\ \frac{1}{j(1+j)} = -0.5 - 0.5j, & k = 1 \\ -\frac{1}{j(1-j)} = -0.5 + 0.5j, & k = -1 \\ 0.1 - 0.3j, & k = 3 \\ 0.1 + 0.3j, & k = -3 \\ 0, & \text{otherwise} \end{cases}
$$

Grading comments: There are some other ways of finding coefficient X_k.

- Apply Lapalce to signal component of $x(t)$ **within one period**, defined as $x_0(t)$ and use

$$
\begin{aligned}
X_k =& \frac{1}{T_0}\mathcal{L}\{x_0(t)\}|_{s=jk\Omega_0} \\
=& \frac{1}{2\pi}\mathcal{L}\{x_0(t)\}|_{s=jk} \\
=& \frac{1}{2\pi}\mathcal{L}\left\{[1+2\sin(t)+2\cos(3t)]\left[u(t)-u(t-2\pi)\right]\right\}|_{s=jk} \\
=& \frac{1}{2\pi}\left[\frac{1}{s}+\frac{2}{s^2+1}+\frac{2s}{s^2+9}\right]\left[1-e^{-2\pi s}\right]|_{s=jk} \\
=& \frac{1}{2\pi}\left[\frac{1}{jk}+\frac{2}{1-k^2}+\frac{2jk}{9-k^2}\right]\left[1-e^{-j2\pi k}\right]
\end{aligned}
$$

One can find $X_k = 0$ for $k \neq 0, \pm 1, \pm 3$ because of the $e^{-j2\pi k}$ term. Otherwise it involves with $\frac{0}{0}$ terms and requires special attention. For example when $k = 3$ the $\frac{0}{0}$ term comes from

$$
X_3 = \frac{1}{2\pi}\frac{2jk(1-e^{-j2\pi k})}{9-k^2}|_{k=3}
$$

Using L'hospital's rule in terms of k it becomes

$$
X_3 = \frac{1}{\pi}\frac{jk(j2\pi)}{-2k} = 1
$$

A **common mistake** is to directly apply Laplace transform to $x(t)$ rather than $x_0(t)$. Besides, since $x(t)$ is not a causal signal, one cannot find Laplace transform of $x(t)$ using single-sided Laplace transform table. Actually Laplace of periodic signal does not converge.

- Compute coefficient using definition

$$
\begin{aligned}
X_k =& \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jkt}dt \\
=& \frac{1}{2\pi}\int_0^{2\pi} \left[1+2\sin(t)+2\cos(3t)\right]e^{-jkt}dt \\
=& \frac{1}{2\pi}\int_0^{2\pi} e^{-jkt}dt + \frac{1}{2j\pi}\int_0^{2\pi} e^{-j(k-1)t}dt - \frac{1}{2j\pi}\int_0^{2\pi} e^{-j(k+1)t}dt \\
&+ \frac{1}{2\pi}\int_0^{2\pi} e^{-j(k-3)t}dt + \frac{1}{2\pi}\int_0^{2\pi} e^{-j(k+3)t}dt
\end{aligned}
$$

Note that all the above intergral has form $\int_0^{2\pi} e^{-jmt}dt$ for some interger m and

$$\int_0^{2\pi} e^{-jmt}dt = \begin{cases} e^{-j2\pi m} - 1 = 0, & m \neq 0 \\ 2\pi, & m = 0 \end{cases}$$

One can find $X_k = 0$ unless $k = 0, \pm 1, \pm 3$. This approach gives same results as in other methods.

(d) The power of output signal is

$$\begin{aligned} P_z = &\sum_{k=-\infty}^{\infty} |Z_k|^2 \\ =&1 + |0.5 + 0.5j|^2 + |0.5 - 0.5j|^2 + |0.1 - 0.3j|^2 + |0.1 + 0.3j|^2 \\ =&1 + 0.5 + 0.5 + 0.1 + 0.1 \\ =&2.2 \end{aligned}$$