EE 102: Midterm Solution – Spring 2012

Problem 1:

$$y(t) = \int_{-\infty}^{\infty} t(t-\tau)u(t-\tau)\, x(\tau)d\tau$$

i) Written as a convolution integral, therefore $h(t,\tau) = t(t-\tau)u(t-\tau)$

When $x(t) = U(t)U(3-t)$, we have

$$t < 0 : y(t) = \int_{-\infty}^{t} t(t-\tau)\cdot 1 \cdot 0 d\tau = 0$$

$$0 \leq t \leq 3: y(t) = \int_{-\infty}^{0} t(t-\tau)\cdot 1 \cdot 0\, d\tau + \int_{0}^{t} t(t-\tau)\cdot 1 \cdot 1\, d\tau = 0 + \frac{t^3}{2}$$

$$t > 3 : y(t) = 0 + \int_{0}^{3} t(t-\tau)\cdot 1 \cdot 1\, d\tau + 0 = 3t^2 - \frac{9t}{2}$$

ii) TV, C

Problem 2:

i) S_1 written as a convolution integral

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}U(t-\tau)x(\tau)d\tau$$

Therefore $h_1(t) = e^{-t}U(t)$.

S_2 written as a convolution integral

$$w(t) = \int_{-\infty}^{\infty} [\delta(t-\tau) - U(t-\tau)]v(\tau)d\tau$$

Therefore $h_2(t) = \delta(t) - U(t)$.

$$h_{21}(t) = \int_{-\infty}^{\infty} h_1(t-\tau)h_2(\tau)d\tau = \int_{0}^{t} e^{-(t-\tau)}[\delta(\tau) - U(\tau)]d\tau = (2e^{-t} - 1)U(t)$$

ii) Since both systems can be expressed in the convolutional integral form with impulse responses of the form $\mathrm{h}(\mathrm{t}-\tau)$ then both systems are LTI. Also, since their $h(t) > 0$ only for $t \geq 0$ then the systems are causal. So both systems are LTIC.

Problem 3:

i) $$H(s)=\frac{L_s\{e^{-t}t-e^{-t}t^2\}}{L_s\{e^{-t}t\}}=\frac{\frac{1}{(s+1)^2}-\frac{2}{(s+1)^3}}{\frac{1}{(s+1)^2}}=\frac{s-1}{s+1}=1-\frac{2}{s+1}$$

$$h(t)=\delta(t)-2e^{-t}U(t)$$

ii) $$\frac{Y(s)}{X(s)}=\frac{L_s\{\cos(t)U(t)\}}{X(s)}=H(s)=\frac{s-1}{s+1}\rightarrow X(s)=\frac{s}{s^2+1}\cdot\frac{s+1}{s-1}$$

$$\frac{s^2+s}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Cs+D}{s^2+1}=\frac{1}{s-1}+\frac{Cs+D}{s^2+1}=\frac{(1+C)s^2+(D-C)s-D+1}{(s-1)(s^2+1)}$$

$$D=1, C=0$$

$$X(s)=\frac{1}{s-1}+\frac{1}{s^2+1}\rightarrow x(t)=(e^t+\sin(t))U(t)$$

Problem 4:

$$L_s\{e^{-t}\{\int_0^t \cos(t-\tau)\,e^{-\tau}\sin(\tau)\,d\tau\}\}=L_s\{e^{-t}g(t)\}$$

$$L_s\{g(t)\}=L_s\{cost(t)U(t)\}\cdot L_s\{e^{-\tau}\sin(\tau)\,U(\tau)\}=\frac{s}{s^2+1}\cdot\frac{1}{(s+1)^2+1}$$

$$L_s\{e^{-t}g(t)\}=\frac{s+1}{(s+1)^2+1}\cdot\frac{1}{(s+2)^2+1}$$

Problem 5:

There is no unique solution to this problem. The problem could be solved using Laplace transforms by writing $F(s)=X(s)H(s)$ for any $X(s)$ and $H(s)$ of your choice, which results in

$$f(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau$$

Another simple way to solve this problem is to convolve the given $f(t)$ with an impulse function by taking advantage of the sifting property,

$$f(t)=\int_{-\infty}^{\infty}(\cos(\tau)+\sin(\tau)-1)U(\tau)\delta(t-\tau)d\tau$$

Problem 6:

Impulse Response:

$$L_s\left\{\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+y(t)=\frac{dx(t)}{dt}+x(t)\right\}Y(s)(s^2+s+1)=X(s)(s+1)H(s)=\frac{Y(s)}{X(s)}$$
$$=\frac{s+1}{s^2+s+1}$$

$$H(S) = \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{\sqrt{3}}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$h(t) = e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]U(t)$$

Unit step response:

$$G(s) = \frac{H(s)}{s} = \frac{s+1}{s(s^2+s+1)}$$

$$G(s) = \frac{1}{s} - \frac{s}{s^2+s+1}$$

$$g(t) = U(t) - e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]U(t)$$

Problem 7:

In convolutional form

$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)}U(t-\tau)x(\tau)d\tau + \int_{-\infty}^{\infty} U(t-\tau)x(\tau)d\tau$$

$$h(t) = (e^{-t}+1)U(t)$$

$$H(s) = \frac{1}{s+1}+\frac{1}{s} = \frac{2s+1}{s^2+s} = \frac{Y(s)}{X(s)}$$

$$Y(s)(s^2+s) = X(s)(2s+1)$$

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt} = 2\frac{dx(t)}{dt}+x(t)$$