UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Examination
Date: March 18, 2019, Duration: 3 hours

INSTRUCTIONS:

- The exam has 6 problems and 17 printed pages.
- The exam is closed-book.
- Two double-sided cheat sheets of A4 size are allowed.
- Calculator is NOT allowed.

Your name:________________________________

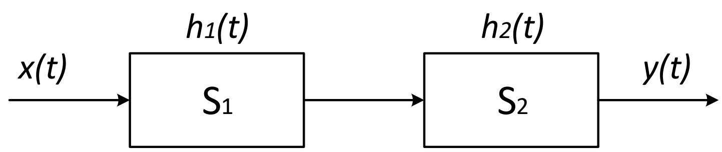
Student ID:________________________________

Table 1: Score Table

Problem	a	b	c	d	Total	Score
1	5	4	6		15	
2	2	2	4	7	15	
3	3	7			10	
4	2	5	1	2	10	
5	6	9			15	
6	6	3	4	2	15	
Sum					80	

1 Problem 1 (15 pts)

Consider a cascade of the systems S_1 and S_2.



The input $x(t)$ and the output $y(t)$ of the cascade are given as follows

$$x(t) = \frac{7}{5}e^{-2t}u(t) - \frac{2}{5}\cos(t)u(t) + \frac{4}{5}\sin(t)u(t)$$

$$y(t) = \frac{1}{2}e^{t-1}u(t-1) - \frac{1}{2}\cos(t-1)u(t-1) - \frac{1}{2}\sin(t-1)u(t-1).$$

(a) (5 pts) Find the transfer function $H(s)$ of the cascade and sketch its zero-pole plot. Denote the region of convergence in the sketch.

(b) (4 pts) If the impulse response function of the system S_2 is

$$h_2(t) = e^t u(t),$$

find the impulse response function $h_1(t)$ of the system S_1.

(c) (6 pts) Find the response $y(t)$ of the cascaded system if the input is $x(t) = e^{4-2t}u(t-2)$.

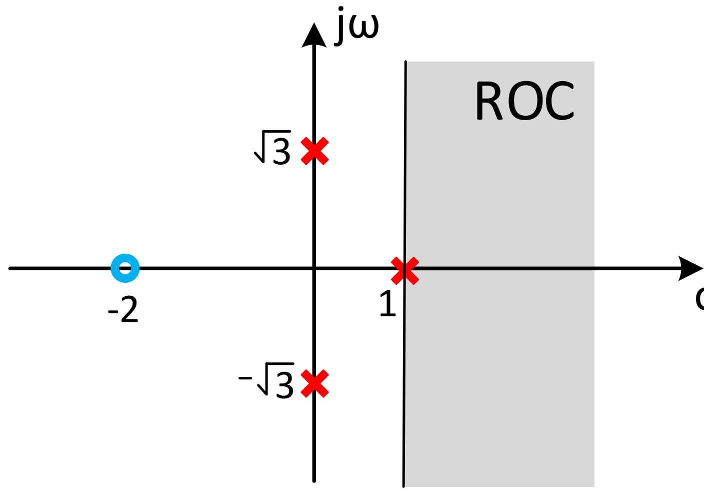
Solution:

(a) Since the input $x(t)$ and the output $y(t)$ are known, we can use their Laplace transforms to find the transfer function $H(s)$.

$$\begin{aligned} X(s) &= \frac{7}{5}\frac{1}{s+2} - \frac{2}{5}\frac{s}{s^2+1} + \frac{4}{5}\frac{1}{s^2+1} \\ &= \frac{s^2+3}{(s^2+1)(s+2)} \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right)e^{-s} \\ &= \frac{1}{(s^2+1)(s-1)}e^{-s} \end{aligned}$$

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ &= \frac{s+2}{(s^2+3)(s-1)}e^{-s} \end{aligned}$$



(b) Given $h_2(t)$, we can calculate the transfer function of the system S_2.

$$H_2(s) = \frac{1}{s-1}.$$

Then $H_1(s)$ can be found in the following way

$$\begin{aligned} H_1(s) &= \frac{H(s)}{H_2(s)} \\ &= \frac{s+2}{s^2+3}e^{-s} \end{aligned}$$

The impulse response function $h_1(t)$ of the system S_1 is

$$h_1(t) = \cos(\sqrt{3}(t-1))u(t-1) + \frac{2}{\sqrt{3}}\sin(\sqrt{3}(t-1))u(t-1),$$

(c) Given the input $x(t)$, we can find its Laplace transform $X(s)$.

$$X(s) = \frac{1}{s+2}e^{-2s}.$$

Then we can find the Laplace transform $Y(s)$ of the output $y(t)$.

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{s+2}{(s^2+3)(s-1)}\frac{1}{s+2}e^{-3s} \\ &= \frac{1}{(s^2+3)(s-1)}e^{-3s} \end{aligned}$$

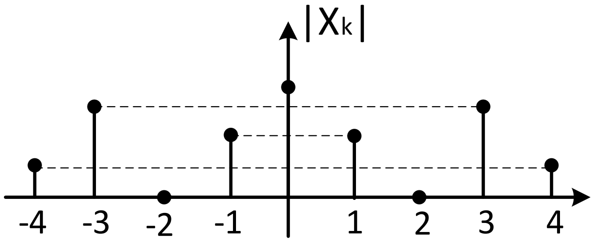
We can decompose $Y(s)$ into partial fractions to find $y(t)$.

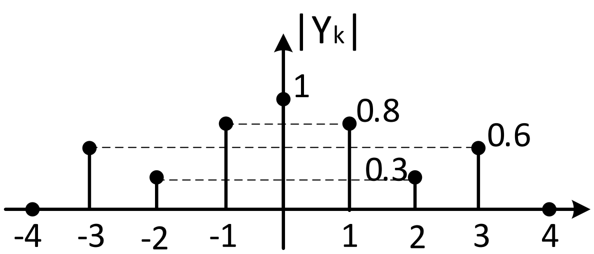
$$\begin{aligned} Y(s) &= \frac{1}{4}\left(-\frac{s+1}{s^2+3}+\frac{1}{s-1}\right)e^{-3s} \\ &= \frac{1}{4}\left(-\frac{s}{s^2+3}-\frac{1}{s^2+3}+\frac{1}{s-1}\right)e^{-3s} \end{aligned}$$

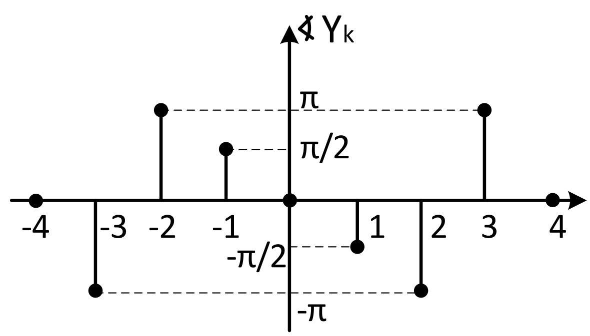
$$y(t) = \frac{1}{4}\left(-\cos(\sqrt{3}(t-3)) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}(t-3)) + e^{t-3}\right)u(t-3)$$

Problem 2 (15 pts)

Consider a system S with the periodic input $x(t)$ and the periodic output $y(t)$. The fundamental period of $x(t)$ is $T_0 = 4$. Let X_k and Y_k be the Fourier series coefficients of $x(t)$ and $y(t)$, respectively. The magnitude spectrum of X_k and the magnitude and phase spectra of Y_k are depicted in the following figures.







(a) (2 pts) Is the system S linear? Explain your answer.

(b) (2 pts) Is the output $y(t)$ a real signal? Explain your answer.

(c) (4 pts) Is the output $y(t)$ even, odd, or neither of these? Explain your answer.

(d) (7 pts) The periodic signal $z(t)$ is obtained by passing the signal $y(t)$ through a filter with the following impulse response function

$$h(t) = \delta(t) - \frac{\sin(\frac{3\pi}{4}t)}{\pi t}.$$

Let Z_k be the Fourier series coefficients of $z(t)$.
Find Z_k and sketch its the magnitude and the phase spectra. Find $z(t)$.

Solution:

(a) The system is not linear since it introduces new frequencies in the output. The input $x(t)$ does not have frequencies that correspond to $\omega = 2\omega_0$ and $\omega = -2\omega_0$ which occur in the output.

(b) The output is a real signal. We can see that from the magnitude and phase spectra of Y_k. The magnitude spectra is an even function, and the phase spectra is an odd function.

(c) The signal $y(t)$ is neither even nor odd. We can see that from the phase spectrum of Y_k. Coefficients for $k = \pm 1$ correspond to a sine in $y(t)$, while coefficients for $k = \pm 2, \pm 3$ correspond to two cosines in $y(t)$. The sum of cosines and sines results in a signal that is neither even nor odd.

Alternatively, we can first show that the signal $y(t)$ is not even by observing that $Y_k \neq Y_{-k}$ for $k = 1$. Then we observe that $y(t)$ is not odd either since $Y_0 \neq 0$. Therefore, the signal $y(t)$ is neither even nor odd.

(d) First, we find the frequency response of the filter.

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\delta(t) - \frac{\sin(\frac{3\pi}{4}t)}{\pi t}\} \\ &= 1 - rect(\omega, \frac{3\pi}{4}). \end{aligned}$$

We can see that $H(\omega)$ is a real function. Moreover, we observe that the filter is a high pass filter with cut-off frequency $w_c = \frac{3\pi}{4}$.

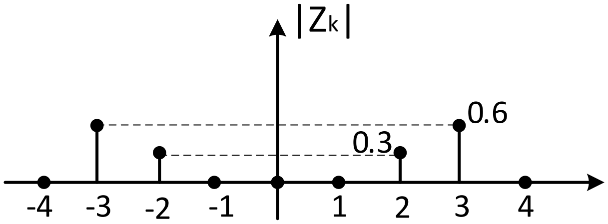
$$H(\omega) = \begin{cases} 1, & |\omega| \geq \frac{3\pi}{4} \\ 0, & |\omega| < \frac{3\pi}{4} \end{cases}$$

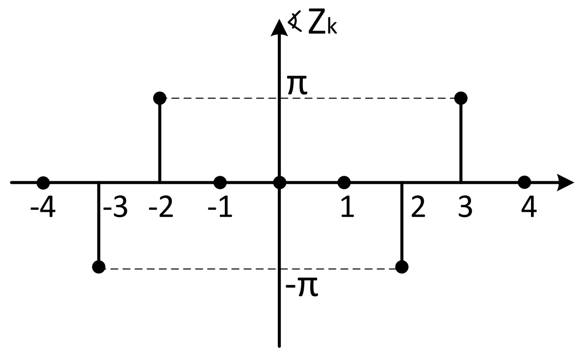
We know that the Fourier coefficients Z_k are obtained after the coefficients Y_k are passed through the filter. The filter is going to remove frequencies lower than $w_c = \frac{3\pi}{4}$, which means that coefficients Y_k for $|k| \leq 1$ are going to be removed. We conclude that the coefficients Z_k are

$$Z_k = \begin{cases} Y_k, & k = \pm 2, \pm 3 \\ 0, & \text{otherwise} \end{cases}$$

The signal $z(t)$ is

$$\begin{aligned} z(t) &= \sum_{k=-\infty}^{\infty} Z_k e^{jk\frac{\pi}{2}t} \\ &= 0.3e^{j\pi}e^{-j2\frac{\pi}{2}t} + 0.3e^{-j\pi}e^{j2\frac{\pi}{2}t} + 0.6e^{j\pi}e^{-j3\frac{\pi}{2}t} + 0.6e^{-j\pi}e^{j3\frac{\pi}{2}t} \\ &= -0.6\cos(\pi t) - 1.2\cos(\frac{3\pi}{2}t) \end{aligned}$$





Problem 3 (10 pts)

Consider a periodic signal with period $T = 2\pi$ defined over one period as follows

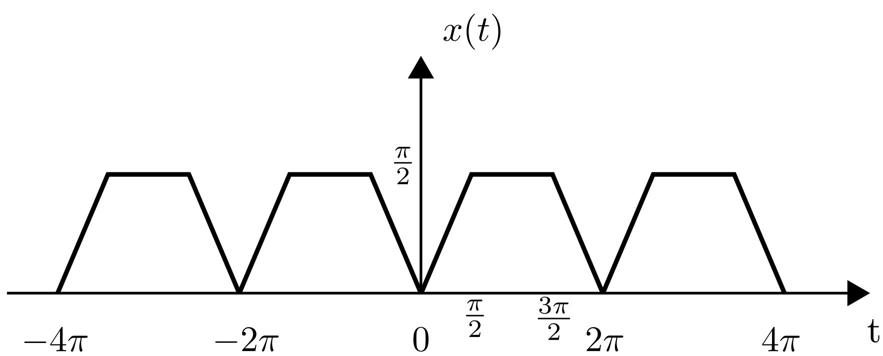
$$x(t) = r(t) - r(t - \frac{\pi}{2}) - r(t - \frac{3\pi}{2}) + r(t - 2\pi) \quad 0 \le t \le 2\pi$$

(a) (3 pts) Plot this periodic signal.

(b) (7 pts) Express this signal in terms of the complex Fourier series.

Solution:

(a) Plotting the signal, we get



(b)

We have $\omega_0 = \frac{2\pi}{2\pi} = 1$. By calculating the area under the figure, we get

$$\begin{aligned}
X_0 &= \frac{1}{T}\int_0^{2\pi} x(t)dt \\
&= \frac{1}{2\pi}\frac{\pi}{2}(\frac{1}{2}\frac{\pi}{2} + \pi + \frac{1}{2}\frac{\pi}{2}) \\
&= \frac{3\pi}{8}
\end{aligned}$$

We calculate X_k using Laplace transform of one period

$$X(s) = \frac{1}{s^2}(1 - e^{-\frac{\pi}{2}s} - e^{-\frac{3\pi}{2}s} + e^{-2\pi s})$$

Then, substituting for $s = j\omega_0 k = jk$ and dividing by 2π we get

$$\begin{aligned}
X_k &= \frac{-1}{2\pi k^2}(1 - e^{-j\frac{\pi}{2}k} - e^{-j\frac{3\pi}{2}k} + e^{-j2\pi k}) \\
&= \frac{-1}{2\pi k^2}(2 - e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k})
\end{aligned}$$

where we used the fact that $e^{-2j\pi k} = 1$. We can then express the signal as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkt}$$

Problem 4 (10 pts)

Consider the following signal

$$x(t) = 1 + cos(t)$$

It is used as an input to the following two systems.

$$S_1 : y(t) = 2x(t)$$
$$S_2 : z(t) = (x(t))^2$$

Let X_k,Y_k, and Z_k be the Fourier series coefficients of $x(t)$,$y(t)$, and $z(t)$ respectively.

(a) (2 pts) Calculate X_k and plot its magnitude spectrum.

(b) (5 pts) Calculate Y_k and Z_k. Plot the magnitude spectrum for each of them.

(c) (1 pts) Can you obtain Y_k from X_k? If yes, find the relation between Y_k and X_k, else justify why it is not possible.

(d) (2 pts) Can you obtain Z_k from X_k? If yes, find the relation between Z_k and X_k, else justify why it is not possible.

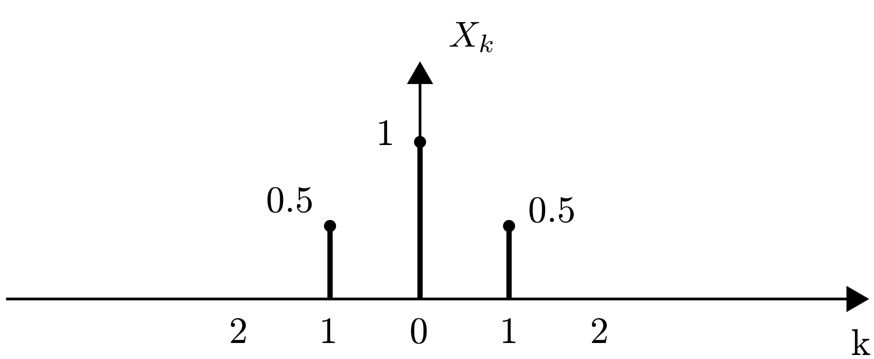
Solution:

(a) $x(t)$ can be rewritten as follows

$$x(t) = 1 + 0.5e^{jt} + 0.5e^{-jt}$$

From which,

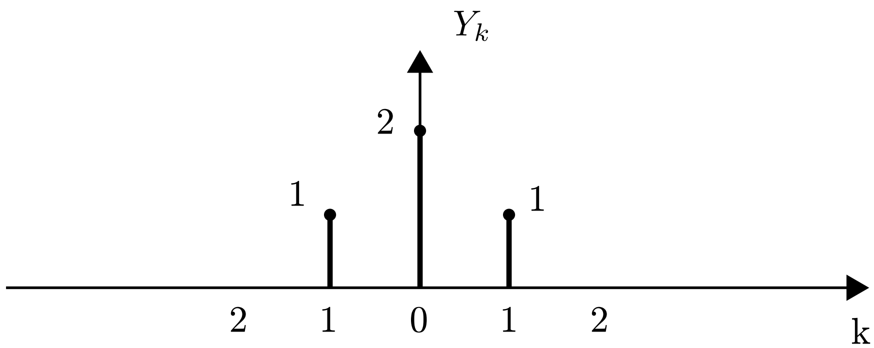
$$X_k = \begin{cases} 1 & k = 0 \\ 1/2 & k = -1, 1 \\ 0 & \text{OW} \end{cases}$$



(b) By applying the system relations we can find that

$$y(t) = 2 + 2cos(t)$$

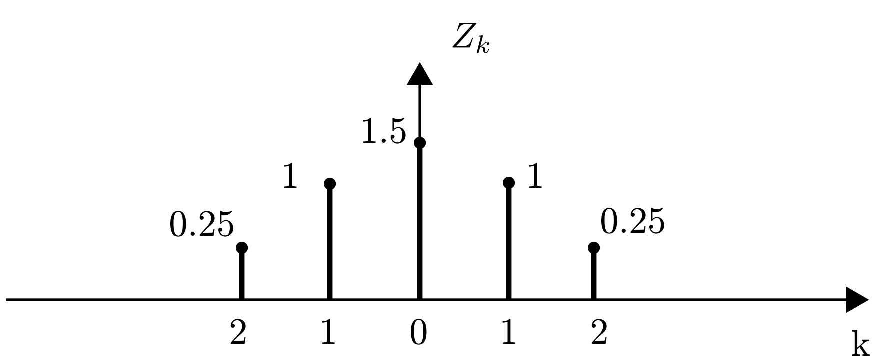
$$Y_k = \begin{cases} 2 & k = 0 \\ 1 & k = -1, 1 \\ 0 & \text{OW} \end{cases}$$



We do the same for $z(t)$

$$\begin{aligned} z(t) &= (1 + cos(t))^2 = 1 + 2cos(t) + cos^2(t) \\ &= 1 + 2cos(t) + \frac{1}{2}(1 + cos(2t)) \\ &= 1.5 + 2cos(t) + \frac{1}{2}cos(2t) \\ &= 1.5 + e^{jt} + e^{-jt} + 0.25e^{j2t} + 0.25e^{-j2t} \end{aligned}$$

$$Z_k = \begin{cases} 1.5 & k = 0 \\ 1 & k = -1, 1 \\ 0.25 & k = -2, 2 \\ 0 & \text{OW} \end{cases}$$



(c) Yes, $Y_k = 2X_k$

(d) No. Because the system is non linear. It introduces new frequency components.

Problem 5 (15 pts)

Let the signal y(t) be defined in the following way

$$y(t) = \big(x(t)\cos^2(t)\big) * \left(\frac{\sin(t)}{\pi t}e^{j2t}\right),$$

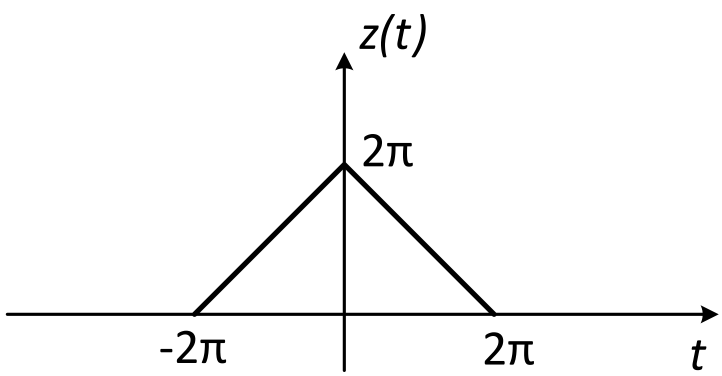
where $*$ denotes convolution. Let $X(\omega)$ and $Y(\omega)$ be the Fourier transforms of $x(t)$ and $y(t)$, respectively. Assume that $x(t)$ is real and $X(\omega) = 0$ for $|\omega| \geq 1$.

(a) (6 pts) Show that $Y(\omega)$ can be expressed in terms of $X(\omega)$. Put the result in the simplest possible form.

(b) (9 pts) Let the signal $x(t)$ be

$$x(t) = z(t) * \frac{\sin(t)}{\pi t},$$

where $z(t)$ is given as in the following figure.



Find the expression for $Y(\omega)$, and sketch the magnitude and phase spectra of $Y(\omega)$. **Do not forget to include values on both axes.**

Solution:

(a) We see that the signal $y(t)$ is obtained by convolution of $x(t)\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)$ and $\frac{\sin(t)}{\pi t}e^{j2t}$. That means that $Y(\omega)$ can be found as follows

$$Y(\omega) = \mathcal{F}\{x(t)\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\}\mathcal{F}\{\frac{\sin(t)}{\pi t}e^{j2t}\}$$

We use the convolution property to reformulate the first part

$$\begin{aligned}\mathcal{F}\{x(t)\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\} &= \frac{1}{2\pi}X(\omega) * \left(\pi\delta(w) - \frac{\pi}{2}\delta(\omega+2) - \frac{\pi}{2}\delta(\omega-2)\right) \\ &= \frac{1}{4}X(\omega+2) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega-2)\end{aligned}$$

We can use frequency shifting property in the second part to shift the ideal low pass filter to the frequency $\omega = 2$.

$$\mathcal{F}\{\frac{\sin(t)}{\pi t}e^{j2t}\} = rect(w-2,1).$$

We see that the low pass filter is going to save just frequencies around $w = 2$.

$$\begin{aligned} Y(\omega) &= \left(\frac{1}{4}X(\omega+2) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega-2)\right) rect(w-2,1) \\ &= \frac{1}{4}X(\omega-2). \end{aligned}$$

(b) To avoid long computations by definition, we can first observe that $z(t)$ is the result of convolution between two rectangular pulses.

$$z(t) = rect(t,\pi) * rect(t,\pi).$$

Then we know that

$$\begin{aligned} Z(\omega) &= \mathcal{F}\{rect(t,\pi)\}\mathcal{F}\{rect(t,\pi)\} \\ &= \left(2\pi\frac{\sin(\pi\omega)}{\pi\omega}\right)^2 \end{aligned}$$

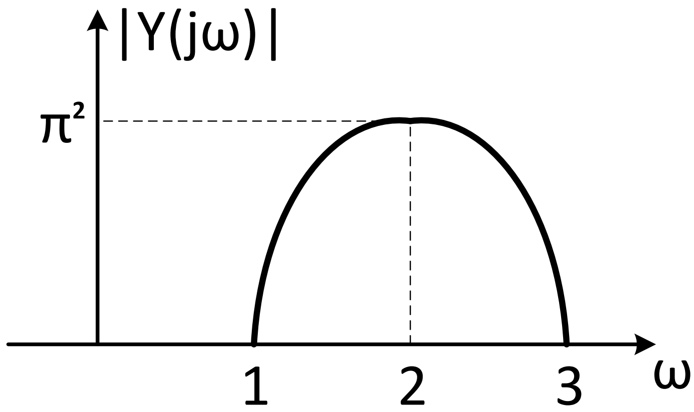
Now we can evaluate $X(\omega)$ in the following way

$$\begin{aligned} X(\omega) &= Z(\omega)rect(w,1) \\ &= \left(2\pi\frac{\sin(\pi\omega)}{\pi\omega}\right)^2 rect(w,1) \\ &= \begin{cases} \left(2\pi\frac{\sin(\pi\omega)}{\pi\omega}\right)^2, & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

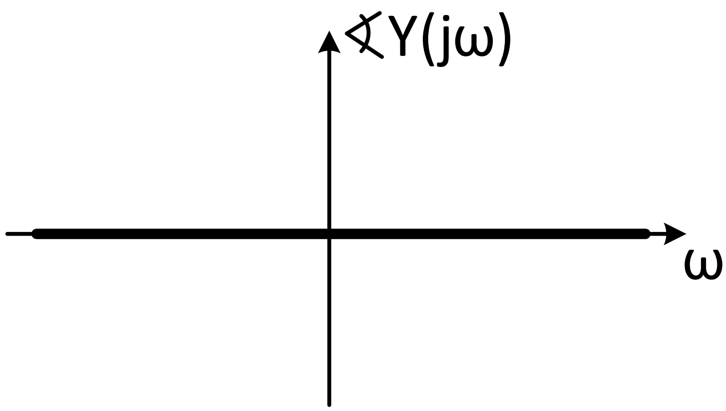
From part a), we know that $Y(\omega) = \frac{1}{4}X(\omega-2)$, thus

$$Y(\omega) == \begin{cases} \left(\pi\frac{\sin(\pi(\omega-2))}{\pi(\omega-2)}\right)^2, & 1 \leq \omega \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

The magnitude spectrum is non-zero in the range $\omega \in (1,3)$.



The phase spectrum is zero for all values of ω since $Y(\omega)$ is a non-negative real function.



Problem 6 (15 pts)

Filtering is used in communication systems to reduce the magnitude of the noise with respect to the signal. Consider the transmitted signal $x(t)$, given by

$$x(t) = \frac{\sin(11t)}{\pi t}$$

The noise signal is given by

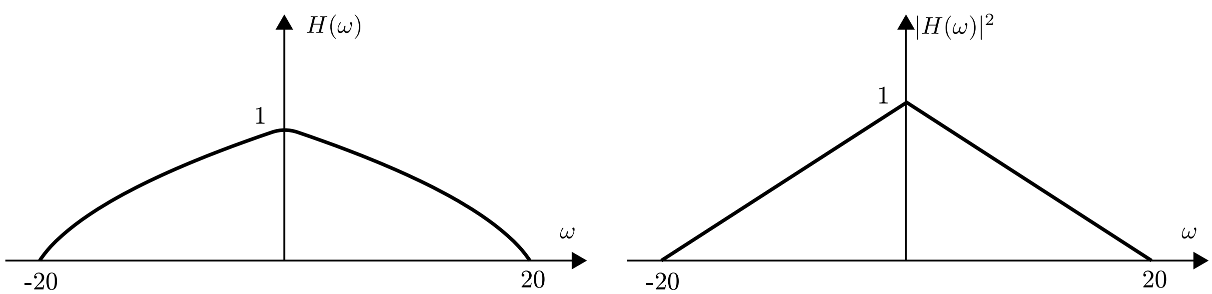
$$n(t) = \frac{\sin(0.5t)}{\pi t} \times 2\cos(19.5t)$$

The received signal is given by

$$y(t) = x(t) + n(t)$$

The Fourier transform of the filter is given by

$$H(\omega) = \frac{1}{\sqrt{20}}\sqrt{20 - |\omega|}(u(\omega + 20) - u(\omega - 20)),$$



The received signal $y(t)$ is then passed through the filter $H(\omega)$, and the result is the signal $z(t)$.

(a) (6 pts) Calculate $X(\omega)$ and $N(\omega)$, the Fourier transform of $x(t)$ and $n(t)$ respectively. Plot their magnitude spectra.

(b) (3 pts) Calculate the energy of $x(t)$ and $n(t)$. Calculate the ratio between the energies of the signal and the noise.

(c) (4 pts) Plot the magnitude spectrum of the output $Z(\omega)$ and on the plot identify the signal and noise components

(d) (2 pts) Calculate the energy of the filtered signal and the filtered noise components. Calculate the ratio between them. Is there an improvement in the ratio of signal to noise energy after applying the filter?

Solution:

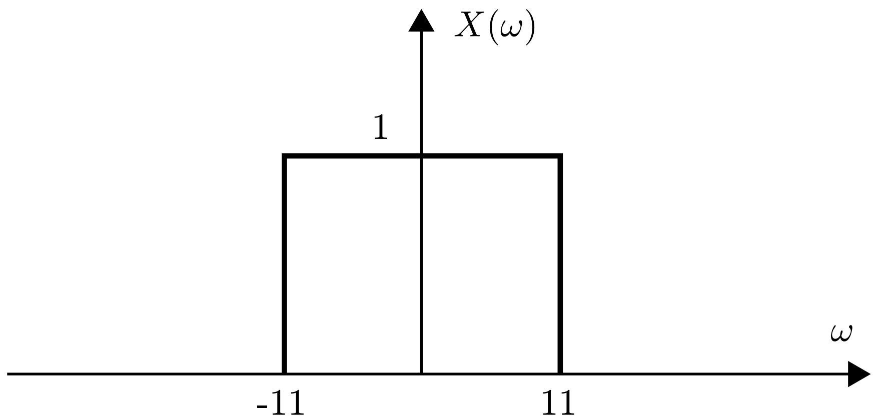
(a) Using Fourier transform table

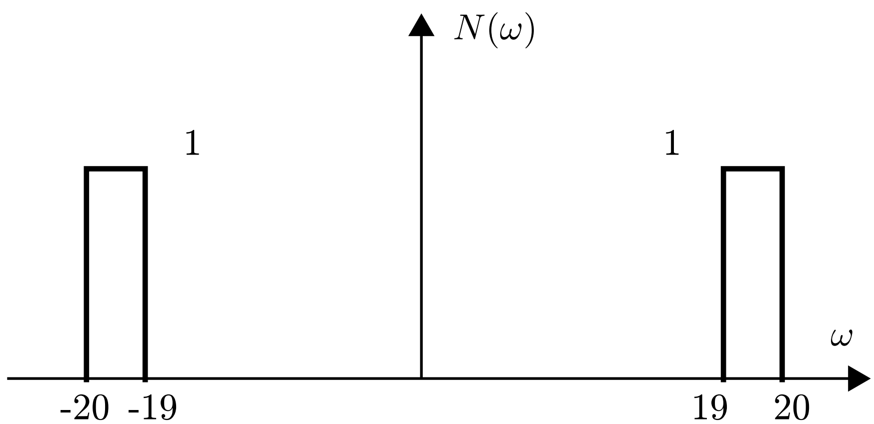
$$X(\omega) = u(\omega + 11) - u(\omega - 11)$$

Using modulation property, we find that

$$N(\omega) = u(\omega + 20) - u(\omega + 19) + u(\omega - 19) - u(\omega - 20)$$

Their magnitude spectrum are as follows

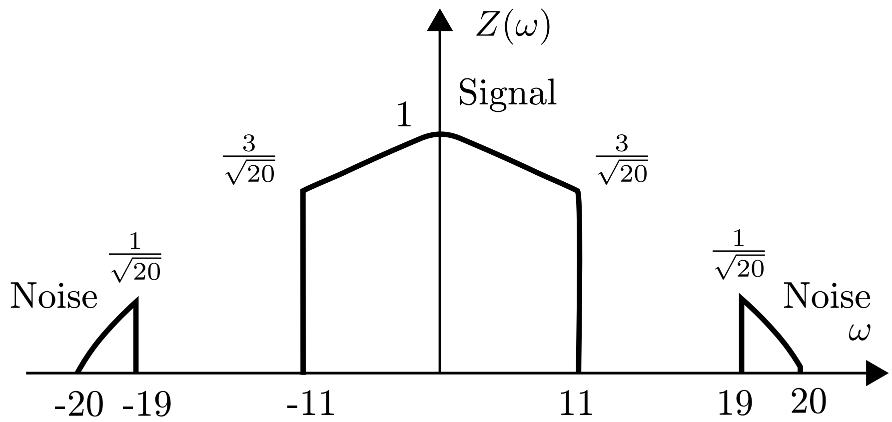


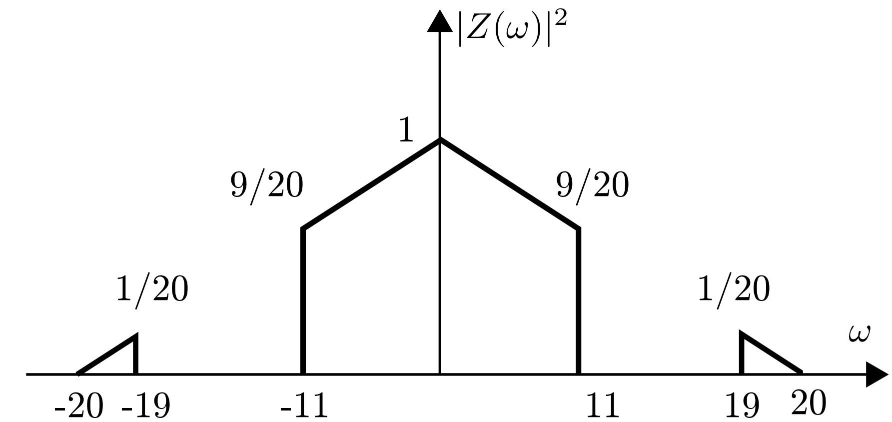


(b) Both $X(\omega)$ and $N(\omega)$ are pure real and since their amplitude is equal one, so, $|X(\omega)|^2$ and $|N(\omega)|^2$ have the same plots as $X(\omega)$ and $N(\omega)$ respectively. By applying parseval's theorem, and calculating the are of the plots, we can find that $E_X = \frac{22}{2\pi}$ and $E_N = \frac{2}{2\pi}$. The signal to noise ratio is 11.

(c) The output can be calculated using

$$Z(\omega) = Y(\omega)H(\omega)$$





(d) We plot $|Z(\omega)|^2$. Using Parseval, the energy of the signal is equal to

$$E_{X_H} = \frac{1}{2\pi} \times 2\frac{(9/20+1)}{2} \times 11 = \frac{319}{40\pi}$$

and the energy of the noise

$$E_{N_H} = \frac{1}{2\pi} \times 2\frac{(1/20 \times 1)}{2} = \frac{1}{40\pi}$$

The ratio between the noise and the signal is 319.

Yes, the signal to noise ratio increased from 11 to 319.