UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Examination
Date: March 22, 2018, Duration: 3 hours

INSTRUCTIONS:

- The exam has 6 problems and 16 pages.
- The exam is closed-book.
- Three double-sided cheat sheets of A4 size are allowed.
- Calculator is NOT allowed.

Your name:____

Student ID:____

Table 1: Score Table

Problem	a	b	c	d	Score
1	3	3	4		10
2	8	2	2		12
3	6	4	4	4	18
4	6	8			14
5	4	4	4	4	16
6	4	4	6	6	20
Total					90

Problem 1 (10 pts) State whether the following statements are TRUE or FALSE. Provide a brief explanation for each part.

(a) (3 pts) A system with the following transfer function is time-variant:

$$h(t,\tau) = e^{-2(t-3)}\cos(6t-3\tau)u(t-3\tau) \tag{1}$$

TRUE, since it cannot be written as function of $t-\tau$ alone

(b) (3 pts) A system with the following input-output relationship is causal:

$$y(t) = x(t) + \int_{t/2}^{2t} e^{-(t-\sigma)}x(\sigma)d\sigma \tag{2}$$

FALSE, since input at $2t$ is required to compute output at t.

(c) (4 pts) If signal $x(t)$ has period T_0 and Fourier series coefficients as follows

$$\begin{aligned} X_2 &= j\pi, \\ X_{-2} &= -j\pi, \\ X_0 &= 1, \\ X_k &= 0, \text{for other values of } k, \end{aligned} \tag{3}$$

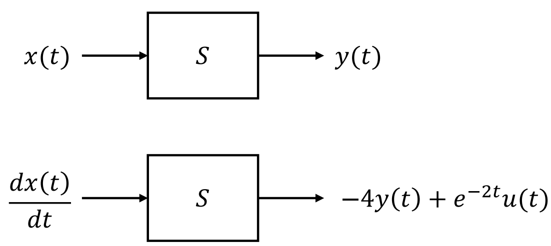
then $x(t)$ is an odd signal.

FALSE, $X_0 \neq 0$, therefore this signal cannot be odd.

Problem 2 (12 pts) Consider an LTI system S with the input signal

$$x(t) = e^{-4t}u(t-1) \tag{4}$$

corresponding output signal $y(t)$. We also know that if input $\frac{dx(t)}{dt}$ is applied to the system S, corresponding output is $-4y(t) + e^{-2t}u(t)$.



(a) (8 pts) Determine the system transfer function $H(s)$ and the impulse response function $h(t)$.

(b) (2 pts) Sketch pole-zero plot of $H(s)$.

(c) (2 pts) Determine if the system is BIBO stable or not.

Solution:

(a) We have $H(s)X(s) = Y(s)$ and $sX(s)H(s) = \left[-4Y(s) + \frac{1}{s+2}\right]$. Substituting the first equation into the second, we get

$$sX(s)H(s) = \left[-4X(s)H(s) + \frac{1}{s+2}\right] \tag{5}$$

$$H(s) = \frac{1}{X(s)(s+4)(s+2)} \tag{6}$$

We have $X(s) = \mathcal{L}[e^{-4t}u(t-1)] = \frac{e^{-(s+4)}}{s+4}$. Substitute in the above equation to get

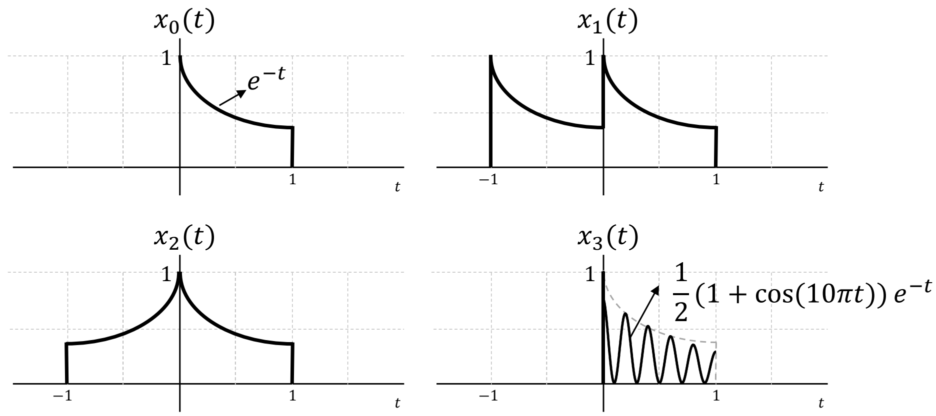
$$H(s) = \frac{e^{s+4}}{(s+2)} \tag{7}$$

$$h(t) = e^4 e^{-2(t+1)}u(t+1) = e^{-2t+2}u(t+1) \tag{8}$$

(b) System has a pole at $s = -2$.

(c) It is BIBO stable since the pole is on the LHS of the $j\Omega$ axis.

Problem 3 (18 pts)
Consider the following real signals



(a) (6 pts) Compute the Fourier transform of $x_0(t)$.

(b) (4 pts) Compute the Fourier transform of $x_1(t)$.

(c) (4 pts) Compute the Fourier transform of $x_2(t)$.

(d) (4 pts) Compute the Fourier transform of $x_3(t)$.

Hint: Parts (b), (c), and (d), can be computed using part (a).
Solution

(a) The Fourier transform of x_0 can be computed as

$$\begin{aligned} X_0(\omega) &= \int_0^1 e^{-t(1+j\omega)}dt \\ &= \left[-\frac{e^{-t(1+j\omega)}}{1+j\omega}\right]_0^1 \\ &= \frac{1-e^{-(1+j\omega)}}{1+j\omega} \end{aligned}$$

(b) Since $x_1(t) = x_0(t) + x_0(t+1)$, we have

$$\begin{aligned} X_1(\omega) &= X_0(\omega) + e^{j\omega} X_0(\omega) \\ &= \frac{1 + e^{j\omega} - e^{-1}(1 + e^{-j\omega})}{1 + j\omega} \end{aligned}$$

(c) Since $x_2(t) = x_0(t) + x_0(-t)$, we have

$$\begin{aligned} X_2(\omega) &= X_0(\omega) + X_0(-\omega) \\ &= \frac{1 - e^{-(1+j\omega)}}{1 + j\omega} + \frac{1 - e^{-(1-j\omega)}}{1 - j\omega} \\ &= \frac{2 - 2e^{-1}\cos(\omega) + 2\omega e^{-1}\sin(\omega)}{1 + \omega^2} \end{aligned}$$

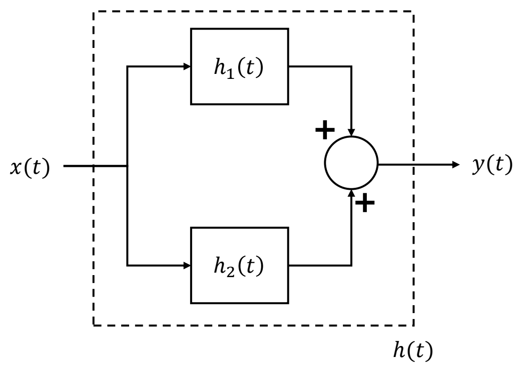
(d) Since $x_3(t) = \frac{1}{2}x_0(t) + \frac{1}{2}\cos(10\pi t)x_0(t)$, we have

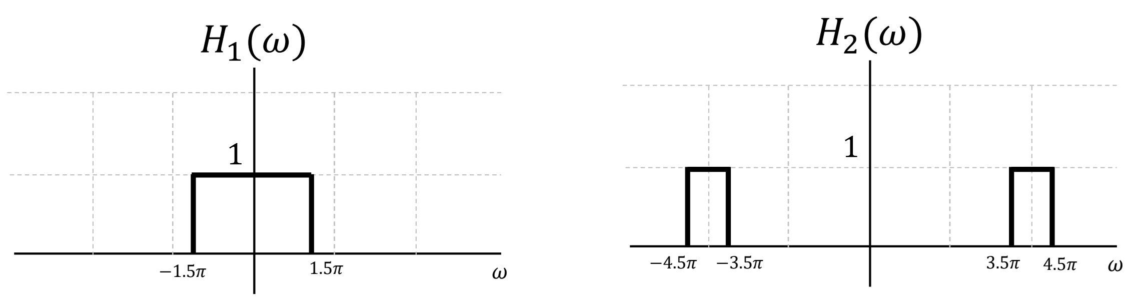
$$\begin{aligned} X_3(\omega) &= \frac{1}{2}X_0(\omega) + \frac{1}{4}\left(X_0(\omega - 10\pi) + X_0(\omega + 10\pi)\right) \\ &= \frac{1}{2}\frac{1 - e^{-(1+j\omega)}}{1 + j\omega} + \frac{1}{4}\frac{1 - e^{-(1+j(\omega-10\pi))}}{1 + j(\omega - 10\pi)} + \frac{1}{4}\frac{1 - e^{-(1+j(\omega+10\pi))}}{1 + j(\omega + 10\pi)} \end{aligned}$$

Problem 4 (16 pts)
Consider a square-wave periodic signal $x(t)$ with the following Fourier Series representation

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{k=1}^{\infty} \frac{\left((-1)^k - 1\right)}{k} \sin(k\pi t)$$

The signal is then passed through a parallel system, with the following frequency response for each branch.





(a) (6 pts) Compute the Fourier transform of $x(t)$.

(b) (2 pts) Sketch the frequency response $H(\omega)$ of the entire system.

(c) (4 pts) Compute the Fourier transform of $y(t)$.

(d) (4 pts) Compute the exponential Fourier series coefficients Y_k, and then find the power of $y(t)$ using Parseval's theorem.

Solution:

(a) Since $b_k = \frac{A}{2\pi k}\big((-1)^k - 1\big)$, and $b_k = \mathsf{Im}\{X_k\}$, then

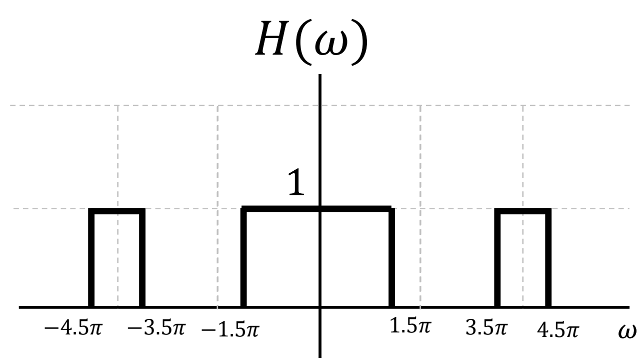
$$X_k = j\frac{A}{2\pi k}\big((-1)^k - 1\big)$$

Also, $X_0 = \frac{A}{2}$. Using the relation between the Fourier series and Fourier transform, we get

$$X(\omega) = A\pi\delta(\omega) + jA\sum_{k=-\infty}^{\infty}\frac{(-1)^k - 1}{k}\delta(\omega - k\pi) \tag{9}$$

where we have used the fact that $\omega_0 = \pi$.

(b) The system's frequency response is $H(\omega) = H_1(\omega) + H_2(\omega)$, which is shown below



(c) The frequency response will pass the DC component of the input, and the components with frequencies between -1.5π and 1.5π, between 3.5π and 4.5π, and between -4.5π and -3.5π. Since $X(\omega)$ is zero at even multiples of π, the output will only have a DC term and two terms at $\pm\pi$. Thus,

$$Y(\omega) = A\pi\delta(\omega) - 2jA\delta(\omega - \pi) + 2jA\delta(\omega + \pi)$$

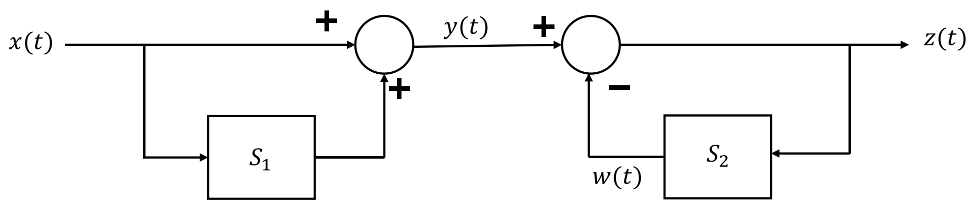
(d) From the relation of the Fourier series and the Fourier transform of periodic signals, we have $Y_k = \frac{1}{2\pi} Y(k\omega_0)$. Thus,

$$Y_k = \begin{cases} \frac{A}{2}, & k = 0 \\ -j\frac{A}{\pi}, & k = 1 \\ j\frac{A}{\pi}, & k = -1 \end{cases}$$

Using Parseval's theorem to compute the power, we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |Y_k|^2 &= \left(\frac{A}{2}\right)^2 + 2\left(\frac{A}{\pi}\right)^2 \\ &= A^2\left(\frac{1}{4} + \frac{2}{\pi^2}\right) \\ &\approx 0.45A^2 \end{aligned}$$

Problem 5 (16 pts) Consider LTI systems S_1 and S_2 arranged as shown in the figure below:



The impulse response function for S_1 is:

$$h_1(t) = \delta(t-1), \tag{10}$$

and the input-output relationship for S_2 is:

$$w(t) = \frac{d^2 z(t)}{dt^2}. \tag{11}$$

The following input is applied to the cascaded system

$$x(t) = \cos\left(\frac{\pi}{2}t\right) + \sin(\pi t). \tag{12}$$

(a) (4 pts) Compute Fourier series coefficients Y_k of $y(t)$.

(b) (4 pts) Plot magnitude and phase spectra of Y_k. Clearly label axes in the plot.

(c) (4 pts) Compute Fourier series coefficients Z_k of $z(t)$.

(d) (4 pts) Plot magnitude and phase spectra of Z_k. Clearly label axes in the plot.
Hint: $\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$, $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$, $\cos(-3\pi/4) = \frac{-1}{\sqrt{2}}$, $\sin(-3\pi/4) = \frac{-1}{\sqrt{2}}$, *Use approximation* $0.5\sqrt{2} \approx 0.7$

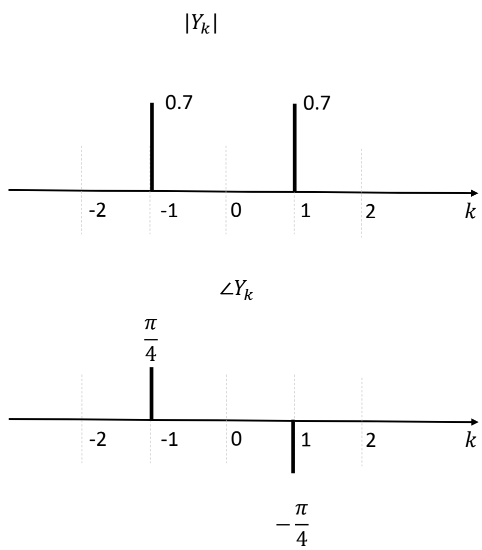
Solution For $x(t) = 0.5e^{j\frac{\pi}{2}t} + 0.5e^{-j\frac{\pi}{2}t} - 0.5je^{j\pi t} + 0.5je^{-j\pi t}$, $\Omega_0 = \pi/2$. Therefore, FS coefficients of $x(t)$ are

$$X_1 = 0.5, X_{-1} = 0.5, X_2 = -0.5j, X_{-2} = 0.5j. \tag{13}$$

(a) $y(t) = x(t) + x(t-1)$. Therefore, FS coefficients are

$$Y_k = X_k + X_k(e^{-jk\Omega_0}) = X_k(1 + e^{-jk\pi/2}) \tag{14}$$
$$Y_1 = 0.5(1-j) = 0.7\angle-\pi/4 \tag{15}$$
$$Y_{-1} = 0.5(1+j) = 0.7\angle\pi/4 \tag{16}$$
$$Y_2 = 0 \tag{17}$$
$$Y_{-2} = 0 \tag{18}$$

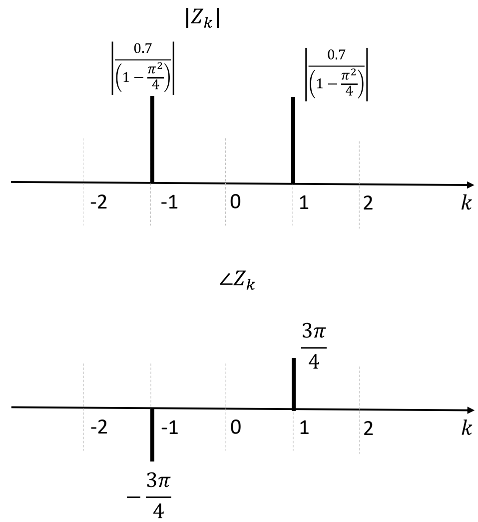
(b) The graph is shown below:



(c) $z(t) = y(t) - \frac{d^2z(t)}{dt^2}$. Therefore,

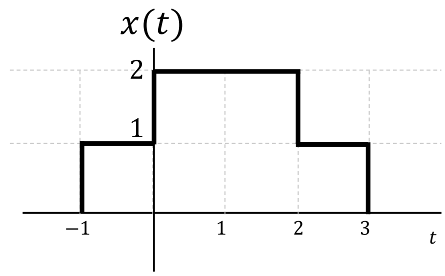
$$Z_k = Y_k - (-k^2\Omega_0^2)Z_k \tag{19}$$
$$Z_k = \frac{Y_k}{\left(1 - k^2\frac{\pi^2}{4}\right)} \tag{20}$$
$$Z_1 = \frac{0.7}{|\left(1 - \frac{\pi^2}{4}\right)|}\angle 3\pi/4 \tag{21}$$
$$Z_{-1} = \frac{0.7}{|\left(1 - \frac{\pi^2}{4}\right)|}\angle - 3\pi/4 \tag{22}$$
$$Z_2 = 0 \tag{23}$$
$$Z_{-2} = 0 \tag{24}$$

(d) The graph is shown below:



Problem 6 (20 pts)

Consider the time-domain real signal $x(t)$ with a Fourier Transform $X(w)$, where $x(t)$ is shown below.



(a) (4 pts) Find $X(0)$ and $\int_{-\infty}^{\infty} X(\omega)d\omega$.

(b) (4 pts) Compute $\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$.

(c) (6 pts) Compute $\int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega$, where $Y(\omega) = \frac{2\sin(\omega)}{\omega}e^{j2\omega}$.
Hint: For any real signals $f(t)$ and $g(t)$, we have

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)d\omega$$

(d) (6 pts) Sketch the inverse Fourier transform of $\mathsf{Real}\{X(\omega)\}$.

Note: You can answer all parts **without** explicitly evaluating $X(w)$.

Solution

(a) Since $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$, then

$$\begin{aligned} X(0) &= \int_{-\infty}^{\infty} x(t)dt \\ &= 6 \end{aligned}$$

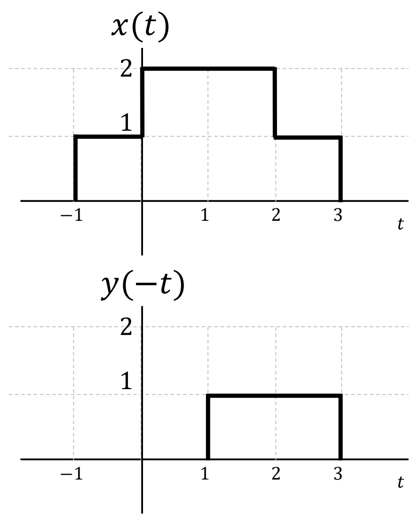
Since $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, then

$$\begin{aligned} \int_{-\infty}^{\infty} X(\omega)d\omega &= 2\pi x(0) \\ &= 4\pi \end{aligned}$$

(b) Using Parseval's theorem $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$, and thus

$$\begin{aligned}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega &= 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\pi(10) \\ &= 20\pi\end{aligned}$$

(c) Let $Z(\omega) = 2\frac{\sin(\omega)}{\omega}$, then the inverse Fourier transform is $z(t) = rect(t)$, i.e., a rectangular function from -1 to 1. Since $Y(\omega) = Z(\omega)e^{2j\omega}$, then $y(t) = z(t+2) = rect(t+2)$, i.e., a rectangular function from -3 to -1. In other words, $\int_{-\infty}^{\infty} x(t)y(-t)dt$ is the area of the product of the following two functions



That is, $\int_{-\infty}^{\infty} x(t)y(-t)dt = 3$. Thus,

$$\int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega = 2\pi \int_{-\infty}^{\infty} x(t)y(-t)dt = 6\pi$$

Alternatively, we can also show that $\int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega$ is actually $2\pi r(0)$, where $r(t) = x(t) * y(t)$. That is, $r(0)$ is the convolution of $x(t)$ and $y(t)$ at $t = 0$, which can be shown to be $r(0) = 3$.

(d) (6 pts) Since $x(t)$ is real, then the inverse Fourier transform of $\mathsf{Real}\{X(\omega)\}$ is the even part of $x(t)$, i.e., $\mathcal{F}^{-1}\{\mathsf{Real}\{X(\omega)\}\} = \frac{x(t)+x(-t)}{2}$. The sketch is provided below.