UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Solutions
Winter Quarter 2016

Question 1

(i) $y(t)$ can also be written as

$$y(t) = \int_{-\infty}^{t-1} e^{\sigma} x(t-\sigma) d\sigma$$

Change of variable $\tau = t - \sigma$ gives

$$\begin{aligned} y(t) &= \int_{1}^{\infty} e^{t-\tau} x(\tau) d\tau \\ &= \int_{-\infty}^{\infty} e^{t-\tau} u(\tau - 1) x(\tau) d\tau \end{aligned}$$

Therefore the IRT is $h(t, \tau) = e^{t-\tau} u(\tau - 1)$.
(ii) System is TV since $h(t, \tau)$ is not a function of $t - \tau$. System is NC since $h(t, \tau)$ is not necessarily zero with $t < \tau$.
(iii)

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{t-\tau} u(\tau - 1) \left[\delta(\tau) + u(\tau - 2)\right] d\tau \\ &= e^{t} \int_{2}^{\infty} e^{-\tau} d\tau \\ &= e^{t-2}. \end{aligned}$$

Question 2

(i)

$$\begin{aligned} x(t) &= t, 0 \leq t \leq 1 \\ &= 1, 1 < t \leq 2 \\ &= 3 - t, 2 < t \leq 3 \end{aligned}$$

Therefore,

$$X(j\omega) = \int_0^1 te^{-j\omega t}dt + \int_1^2 1e^{-j\omega t}dt + \int_2^3 (3-t)e^{-j\omega t}dt$$
$$X(j\omega) = \int_0^1 te^{-j\omega t}dt + \int_1^2 1e^{-j\omega t}dt + \int_2^3 3e^{-j\omega t}dt - \int_2^3 te^{-j\omega t}dt$$

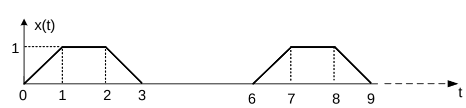
$$\begin{aligned} \int_a^b te^{-j\omega t} &= t\frac{e^{-j\omega t}}{-j\omega}|_a^b + \frac{1}{j\omega}\int_a^b e^{-j\omega t}dt = t\frac{e^{-j\omega t}}{-j\omega}|_a^b + \frac{1}{w^2}e^{-j\omega t}|_a^b \\ &= \frac{be^{-j\omega b} - ae^{-j\omega a}}{-j\omega} + \frac{e^{-j\omega b} - e^{-j\omega a}}{w^2} \end{aligned}$$

$$\int_a^b ce^{-j\omega t}dt = c\frac{e^{-j\omega b} - e^{-j\omega a}}{-j\omega}$$

Therefore, $X(j\omega)$ becomes

$$X(j\omega) = \frac{-e^{-j3w} + e^{-j2w} + e^{-j\omega} - 1}{w^2}$$

(ii)
$y(t)$ repeats periodically with period $T_0 = 6$.



(iii)
The period and fundamental frequency of $y(t)$ are

$$T_0 = 6, \quad w_0 = 2\pi/T_0 = \pi/3.$$

Fourier coefficients are

$$\begin{aligned}
Y_0 =& 1/3,\\
Y_k =& \frac{X(jkw_0)}{T_0}\\
=& \frac{1}{6}\left[\frac{-e^{-j3k\pi/3}+e^{-j2k\pi/3}+e^{-jk\pi/3}-1}{(k\pi/3)^2}\right]\\
=& \frac{3}{2}\left[\frac{(-1)^{k+1}-1+e^{-j2k\pi/3}+e^{-jk\pi/3}}{k^2\pi^2}\right], k \neq 0.
\end{aligned}$$

Question 3

The period and fundamental frequency are

$$T_0 = 4, \quad w_0 = \pi/2$$

We can use fourier series to represent signal $x(t)$ by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkw_0 t}$$

Given $X_k = 0, \forall |k| \geq 2$, the above expression can be reduced to

$$x(t) = X_{-1}e^{-j\omega_0 t} + X_0 + X_1 e^{j\omega_0 t}$$

$X_0 = 1 = \frac{1}{4}\int_0^4 x(t)dt$
using Parseval's theorem, and $X_{-1} = X_1$
$1 + 2|X_1|^2 = 33 = \frac{1}{4}\int_0^4 |x(t)|^2 dt$
$|X_1| = 4$

Using the phase of X_k, $X_1 = 4e^{j\pi/4}, X_{-1} = 4e^{-j\pi/4}$ and

$$x(t) = 4e^{-j\pi/4}e^{-j\frac{\pi}{2}t} + 1 + 4e^{j\pi/4}e^{j\frac{\pi}{2}t} \quad (1)$$

$$x(t) = 1 + 8\cos\left(\frac{\pi}{2}t + \pi/4\right) \quad (2)$$

Question 4

$$y(t) = \int_{-\infty}^{\infty} e^{-3(t-\tau)}\cos(t-\tau)u(t-\tau)x(\tau)dt$$

(i)

$$h(t) = e^{-3t}\cos(t)u(t)$$
$$H(s) = \frac{s+3}{(s+3)^2+1}$$

(ii)
Yes. Because the poles are at $s = -3 \pm j$ and not on $j\omega$ axis.

$$H(j\omega) = \frac{j\omega+3}{(j\omega+3)^2+1}$$

(iii)

$$X(s) = \frac{1}{s+3}$$
$$Y(s) = X(s).H(s) = \frac{1}{(s+3)^2+1}$$
$$y(t) = e^{-3t}\sin(t)u(t)$$

$$Y(j\omega) = \frac{1}{(j\omega+3)^2+1} = \frac{1}{-\omega^2+6j\omega+10}$$
$$Y(j\omega) = \frac{(10-\omega^2)-6j\omega}{(10-\omega^2)^2+36\omega^2}$$
$$|Y(j\omega)| = \frac{1}{\sqrt{(10-\omega^2)^2+36\omega^2}}$$
$$\angle Y(j\omega) = arctan\left(\frac{-6\omega}{10-\omega^2}\right)$$

Question 5

(i)

$$x(t) = \cos(\omega_0 t) = \frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

Using the eigenfunction property, the corresponding output is

$$y'(t) = \frac{e^{j\omega_0 t}H(j\omega_0)+e^{-j\omega_0 t}H(-j\omega_0)}{2}$$

But, the given output is:

$$y(t) = j\sin(\omega_0 t)H(j\omega_0) = \frac{e^{j\omega_0 t}H(j\omega_0) - e^{-j\omega_0 t}H(j\omega_0)}{2}$$

Therefore, the required condition is $H(j\omega_0) = -H(-j\omega_0)$.
(ii) Using the eigenfunction property, output corresponding to input $x_2(t) = j\sin(\omega_0 t)$ is

$$y_2(t) = \frac{e^{j\omega_0 t}H(j\omega_0) - e^{-j\omega_0 t}H(-j\omega_0)}{2}$$

Using the condition $H(j\omega_0) = -H(-j\omega_0)$, the output reduces to

$$y_2(t) = \frac{e^{j\omega_0 t}H(j\omega_0) + e^{-j\omega_0 t}H(j\omega_0)}{2} = \cos(\omega_0 t)H(j\omega_0)$$

Question 6
(i) Yes, both S_1 and S_2 are LTI systems.

$$\begin{aligned} h_1(t) =& e^{-|t|}u(t) = e^{-t}u(t) \\ h_2(t) =& \delta(t-1). \end{aligned}$$

(ii) Since both S_1 and S_2 are LTI systems, $H_{12}(j\omega) = H_1(j\omega)H_2(j\omega)$.

$$\begin{aligned} H_1(j\omega) =& \frac{1}{1+j\omega}, \\ H_2(j\omega) =& e^{-j\omega}, \\ H_{12}(j\omega) =& \frac{e^{-j\omega}}{1+j\omega}. \end{aligned}$$

(iii) The Fourier transform of $x(t)$ is

$$X(j\omega) = U\left(\omega + \frac{\pi}{4}\right) - U\left(\omega - \frac{\pi}{4}\right).$$

Therefore the spectrum of $z(t)$ is

$$Z(j\omega) = \begin{cases} \frac{e^{-j\omega}}{1+j\omega}, & \omega \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \\ 0, & \text{otherwise} \end{cases}$$

The energy of $z(t)$ is

$$E_z = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z(j\omega)|^2 d\omega$$
$$= \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\omega^2} d\omega$$
$$= \frac{1}{2\pi}\left[\tan^{-1}\left(\frac{\pi}{4}\right) - \tan^{-1}\left(-\frac{\pi}{4}\right)\right]$$
$$= \frac{1}{\pi}.$$

Question 7

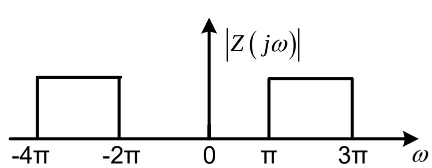
(i) Using the equation of Fourier transform for periodic signal, the answer is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\omega - k\omega_0)$$
$$= 2\pi\delta(\omega) + 4\pi\delta(\omega - 2\pi) + 2\pi e^{j\pi/4}\delta(\omega + 3\pi).$$

(ii) The high-pass filter rejects DC terms in $x(t)$. Thus

$$Y(j\omega) = 4\pi\delta(\omega - 2\pi) + 2\pi e^{j\pi/4}\delta(\omega + 3\pi).$$

(iii) The amplitude spectrum is shown below. Since the signal $z(t)$ is ban-



dlimited within -4π to 3π, the Nyquist frequency is $\omega_s = \frac{2\pi}{T_s} = 7\pi$.

Thus the lowest sampling rate is $1/T_s = 3.5$ Hz.