Practice Final solution, WINTER 2011, EE 102

1. (i)

$$X(s) = 1/s, Z(s) = \frac{s+2}{s^2+1} - \frac{1}{s+1/2} = \frac{5s}{(s^2+1)(2s+1)}.$$

$$H_{12}(s) = \frac{Z}{X} = \frac{5s^2}{(s^2+1)(2s+1)}$$

$$h_1(t) = \delta(t) - \frac{1}{2}e^{-t/2}U(t).$$

$$H_1(s) = 1 - \frac{1}{2s+1} = \frac{2s}{2s+1}$$

$$H_2(s) = \frac{H_{12}(s)}{H_1(s)} = \frac{5s/2}{s^2+1}.$$

$$h_2(t) = \frac{5}{2}\cos tU(t).$$

(ii) No, as the roots are on the $i\omega$ axis.

$$H_2(i\omega) = \frac{5}{2}[\frac{\pi}{2}(\delta(\omega-1) - \delta(\omega+1)) + \frac{i\omega}{1-\omega^2}]$$

$$X(i\omega) = 2\pi\delta(\omega) + i\pi[\delta(\omega+2) - \delta(\omega-2)]$$

$$Y(i\omega) = X(i\omega)H_2(i\omega) = \frac{i\omega 5/2}{1-\omega^2}\left[2\pi\delta(\omega) + i\pi[\delta(\omega+2) - \delta(\omega-2)]\right]$$

$$= \frac{-5/2\pi}{-3}\left[-2\delta(\omega+2) - 2\delta(\omega-2)\right] = \frac{-5}{3}\cos 2t.$$

(iii)

$$v(t) = e^{-2|t|}, S_1 : w(t) = v(t) - \frac{1}{2}\int_{-\infty}^{t} e^{-\frac{t-\sigma}{2}}v(\sigma)d\sigma = e^{-2|t|} - \frac{1}{2}\int_{-\infty}^{t} e^{-2|\tau|}e^{-\frac{(t-\tau)}{2}}d\tau$$

$$= e^{-2|t|} - \frac{1}{2}e^{-t/2}\int_{-\infty}^{t} e^{-2|\tau|}e^{\frac{\tau}{2}}d\tau$$

For $t > 0$:

$$w(t) = e^{-2|t|} - \frac{1}{2}e^{-t/2}\left[\int_{-\infty}^{0} e^{2\tau}e^{\frac{\tau}{2}}d\tau + \int_{0}^{t} e^{-2\tau}e^{\frac{\tau}{2}}d\tau\right] = e^{-2|t|} + \left(\frac{-8}{15}e^{-t/2} + \frac{1}{3}e^{-2t}\right)$$

For $t \leq 0$:

$$w(t) = e^{-2|t|} - \frac{1}{2}e^{-t/2}\int_{-\infty}^{t} e^{2\tau}e^{\frac{\tau}{2}}d\tau = e^{-2|t|} - \frac{1}{5}e^{2t}$$

2.

$$H(s) = \frac{s}{s+1} = 1 - \frac{1}{s+1}, h(t) = \delta(t) - e^{-t}U(t).$$
$$h(t) = \delta(t) - e^{-t}U(t).$$

Let $x_1(t) = e^{-t}U(t-1)$ and $x_2(t) = \cos t$. Let $y_1(t)$ be the output caused by $x_1(t)$ and $y_2(t)$ be the output caused by $\mathrm{x}_2(t)$. $y_1(t)$ can be calculated using Laplace Transform as $h(t)$ and $x_1(t)$ equal 0 for $t < 0$. $y_2(t)$ has to be calculated using Fourier Transform or in time domain as $x_2(t) \neq 0$ for $-\infty < t < 0$.

$$x_1(t) = e^{-1}e^{-(t-1)}U(t-1) \rightarrow X_1(s) = \frac{e^{-1}e^{-s}}{s+1} \rightarrow Y_1(s) = H(s)X_1(s) = \frac{e^{-1}e^{-s}s}{(s+1)^2}.$$

Let $V(s) = \frac{Y_1(s)}{e^{-1}e^{-s}} = \frac{s}{(s+1)^2}$ and $G(s) = \frac{V(s)}{s} = \frac{1}{(s+1)^2}$.

$$G(s) = \frac{1}{(s+1)^2} = \frac{-d}{ds}\frac{1}{s+1}$$

$$\therefore g(t) = te^{-t}U(t)$$

$$L\{\frac{d}{dt}g(t)\} = sG(s) - g(0) = sG(s) = V(s)$$

$$\therefore v(t) = \frac{d}{dt}g(t) = (-te^{-t}+e^{-t})U(t)+te^{-t}\delta(t) = (-te^{-t}+e^{-t})U(t) = e^{-t}(1-t)U(t).$$

$$\therefore y_1(t) = e^{-1}e^{-(t-1)}(1-(t-1))U(t-1) = e^{-t}(2-t)U(t-1).$$

$$X_2(i\omega) = \pi[\delta(\omega-1) + \delta(\omega+1)], H(i\omega) = \frac{i\omega}{1+i\omega}.$$

$$Y_2(i\omega) = \pi[\frac{i}{1+i}\delta(\omega-1) - \frac{i}{1-i}\delta(\omega+1)] = \pi[\frac{1+i}{2}\delta(\omega-1) + \frac{1-i}{2}\delta(\omega+1)]$$

$$\therefore y_2(t) = \frac{\cos t - \sin t}{2}.$$

$$\therefore y(t) = e^{-t}(2-t)U(t-1) + \frac{\cos t - \sin t}{2}.$$

3. Yes.

$$3 \to H(0) = 0 \to 0.$$

$5\cos(20t) \to H(i\omega) \to \sin(20t+\theta)$ is valid, as an LTI system with a sinusoidal input produces a sinusoidal output with the same frequency.

4.

$$\int_{-\infty}^{\infty} x(t)dt = A = X(0), \int_{-\infty}^{\infty} y(t)dt = B = Y(0).$$

$$\int_{-\infty}^{\infty} x(t) * y(t)dt \to Z(0) = \int_{-\infty}^{\infty} z(t)dt$$

$$Z(0) = X(0)Y(0) = AB.$$

5i.

$$F_0 = \frac{1}{T}\int_T f(t)dt = 0.5.$$

$$F_n = \frac{1}{T}\int_T e^{-in\omega_0 t}[\delta(t) - 0.5\delta(t-0.5)]dt = 1 - 0.5e^{-in\omega_0 0.5} = 1 - 0.5(-1)^n.$$

$$x(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty}(1 - 0.5(-1)^n)e^{i2\pi tn}.$$

ii)

$$X(i\omega) = \pi\delta(\omega) + 2\pi\sum_{n=-\infty}^{\infty}(1 - 0.5(-1)^n)\delta(\omega - 2\pi n).$$

iii)

$$i\omega Y(i\omega) + 2Y(i\omega) = 2X(i\omega)$$

$$y(t) = h(t) * x(t) = 2e^{-2t}U(t) * \sum_{-\infty}^{\infty} \delta(t-n) - \frac{1}{2}\delta(t-n-\frac{1}{2})$$

$$= \sum_{-\infty}^{\infty} 2e^{-2(t-n)}U(t-n) - e^{-2(t-n-\frac{1}{2})}U(t-n-\frac{1}{2})$$

$$= e^{-2t}\sum_{-\infty}^{\infty} 2e^{2n}U(t-n) - ee^{2n}U(t-n-\frac{1}{2})$$

For $0 < t < \frac{1}{2}$:

$$y(t) = e^{-2t}\frac{2-e^{-1}}{1-e^{-2}}$$

For $\frac{1}{2} \leq t \leq 1$:

$$y(t) = e^{-2t}\frac{2-e^{1}}{1-e^{-2}}.$$

iv)

$$y(t) = \frac{1}{2} + \sum_{n=-\infty}^{\infty} (1-0.5(-1)^n)e^{i2\pi tn}\frac{2}{2+i2\pi n}.$$

$$MSE = \frac{1}{T}\int_T |y(t)|^2 dt - \sum_{n=-2}^{2} |Y_n|^2,$$

$$\frac{1}{T}\int_T |y(t)|^2 dt = \int_0^{0.5} e^{-4t}\left(\frac{2-e^{-1}}{1-e^{-2}}\right)^2 dt + \int_{0.5}^{1} e^{-4t}\left(\frac{2-e^{1}}{1-e^{-2}}\right)^2 dt = 0.790376,$$

$$\sum_{n=-2}^{2} |Y_n|^2 = 0.67635,$$

$$\therefore MSE = 0.114025.$$

6i.

$$y(t) = \int_{-\infty}^{\infty} [U(\tau - t)U(t+1-\tau) - U(\tau - t + 1)U(t-\tau)]x(\tau)d\tau$$

$$h(t-\tau) = U(\tau - t)U(t+1-\tau) - U(\tau - t + 1)U(t-\tau)]$$

$$h(t) = U(-t)U(t+1) - U(-t+1)U(t)]$$

ii)

$$H(i\omega) = \int_{-1}^{0} e^{-i\omega t} dt - \int_{0}^{1} e^{-i\omega t} dt = \frac{1}{-i\omega}[1 - e^{i\omega}] - \frac{1}{-i\omega}[e^{-i\omega} - 1] = \frac{2}{i\omega}[\cos\omega - 1].$$

iii)

$$y(t) = \int_{t}^{t+1} \cos(\pi\tau - 1)d\tau - \int_{t-1}^{t} \cos(\pi\tau - 1)d\tau = \frac{-4}{\pi}\sin(\pi t - 1)$$

iv)

$$x_2(t) = \sum_{n=-\infty}^{\infty} e^{-|t-n|}\cos(t-n),$$

$$x_2(t+T) = \sum_{n=-\infty}^{\infty} e^{-|t+T-n|}\cos(t+T-n),$$

Let $\tau = n - T$:

$$x_2(t+T) = \sum_{n=-\infty}^{\infty} e^{-|t-\tau|}\cos(t-\tau).$$

Note that the indices of the summation are $n = ..., -2, -1, 0, 1, 2...$, which correspond to $\tau = ..., -2-T, -1-T, -T, 1-T, 2-T, ...$ For $x_2(t)$ to be equal to $x_2(t+T)$, T should be equal to 1.

$$y_2(t) = \int_{t}^{t+1} x_2(\tau)d\tau - \int_{t-1}^{t} x_2(\tau)d\tau$$

Since $x_2(t)$ is periodic with period 1, then $\int_{t}^{t+1} x_2(\tau)d\tau = \int_{t-1}^{t} x_2(\tau)d\tau$.

$$\therefore y_2(t) = 0.$$