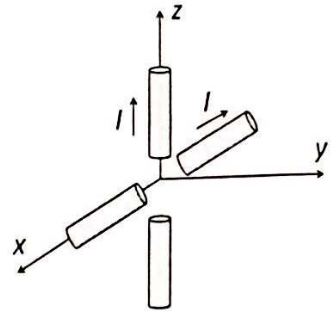
Problem 1: (35 Points)



The far electric field of two Hertzian dipoles at right angles to each other, fed by equal amplitude currents with a 90° phase difference is

$$\vec{E} = \frac{jk\eta IL}{4\pi r} e^{-jkr}[(\sin\theta - j\cos\theta\cos\phi)\hat{\theta} + j\sin\phi\hat{\phi}]$$

Where (r,θ,ϕ) specify the spherical coordinates, k is wave phase constant, I is the antenna current, and L is the antenna length. Find the average power density of antenna radiation, total radiated power, radiation resistance, and directivity of the antenna as a function of I, L, and k.

$$S_{av} = \frac{|\vec{E}|^2}{2\eta} = \frac{|E_\theta|^2 + |E_\phi|^2}{2\eta} = \frac{k^2\eta I^2 L^2}{32\pi^2 r^2}\underbrace{\left[\sin^2\theta + \cos^2\theta\cos^2\phi + \sin^2\phi\right]}_{} = \overbrace{\frac{k^2\eta I^2 L^2}{16\pi^2 r^2}}^{S_{max}} \cdot \overbrace{\frac{1+\sin^2\theta\sin^2\phi}{2}}^{F(\theta,\phi)}$$

$$\hookrightarrow = \sin^2\theta + (1-\sin^2\theta)(1-\sin^2\phi) + \sin^2\phi$$

$$= 1 + \sin^2\theta\sin^2\phi \quad (max = 2 @ \theta = \phi = \tfrac{\pi}{2})$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} r^2 S_{av}(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= r^2 S_{max}\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}(1+\sin^2\theta\sin^2\phi)\sin\theta\, d\theta\, d\phi = r^2 S_{max}\frac{1}{2}\left[\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}\sin^3\theta\sin^2\phi\, d\theta\, d\phi\right]$$

$$= r^2 S_{max}\frac{1}{2}\left[4\pi + \int_0^{2\pi}\left(\int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta\right)\sin^2\phi\, d\phi\right] = r^2 S_{max}\frac{1}{2}\left(4\pi + \int_0^{2\pi}\frac{4}{3}\cdot\frac{1-\cos 2\phi}{2}\, d\phi\right)$$

$$= r^2 S_{max}\frac{1}{2}\left(4\pi + \frac{4}{3}\pi\right) = r^2 S_{max}\cdot\underbrace{\frac{8}{3}\pi}_{= \Omega_p} = \frac{k^2\eta I^2 L^2}{6\pi}$$

$$R_{rad} = \frac{2P_{rad}}{I^2} = \frac{k^2\eta L^2}{3\pi}$$

$\Omega_p = \int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi = \frac{8}{3}\pi$ as derived for P_{rad}. So $D = \frac{4\pi}{\Omega_p} = 1.5$.

or $D = \frac{4\pi r^2 S_{max}}{P_{rad}} = 1.5$ (same as a single dipole!)

Problem 2: (25 points)

A parabolic antenna is designed to have a gain of 30dB at 300MHz.

a) Assuming a radiation efficiency of 55%, find the diameter and estimate the half-power beam width of the antenna.

b) Find the gain and half-power beam width if the antenna is operated at 150MHz.

a) $G = 30\,dB = 1000 \Rightarrow D = \frac{G}{\xi} = \frac{1000}{0.55} \Rightarrow A_e = \frac{\lambda^2 D}{4\pi} = \frac{\left(\frac{3\times10^8}{3\times10^8}\right)^2 \times \frac{1000}{0.55}}{4\pi}$

$= \pi\left(\frac{d}{2}\right)^2$

$\Rightarrow d = 2\sqrt{\frac{1000}{2.2\pi^2}} = 13.57\,m$

$\beta \simeq \frac{\lambda}{d} = \frac{1}{13.57} = 0.0737$ (rad)

b) $D' = \frac{4\pi A_e}{\lambda'^2} = \frac{4\pi \cdot \pi\left(\frac{13.57}{2}\right)^2}{\left(\frac{3\times10^8}{1.5\times10^8}\right)^2} = \pi^2\left(\frac{13.57}{2}\right)^2 = 454$

$G' = \xi D' = 250. \; (= \frac{1}{4}G)$

$\beta' \simeq \frac{\lambda'}{d} = 2\beta = 0.147$ (rad)

Problem 3: (40 Points)

A plane wave in air is incident on a half-wave dipole antenna while antenna axis is oriented along the $(\frac{\sqrt{3}}{2}\hat{y}, \frac{1}{2}\hat{z})$ axis. Assuming the electric field of the plain wave is given by,

$$\tilde{E} = \hat{y}e^{-jx} - \hat{z}e^{-jx} \qquad \text{(V/m)}$$

a) Determine the propagation direction and the average power density of the plane wave.

b) Calculate the induced open circuit voltage at the antenna input port and the maximum absorbed power by the antenna.

c) How can we change antenna orientation to maximize the absorbed power? What would be the induced open circuit voltage and the maximum absorbed power by the antenna at the optimum antenna orientation?

d) Repeat part (b) if the plane wave is incident on a 10 mm long Hertzian dipole antenna oriented along z axis.

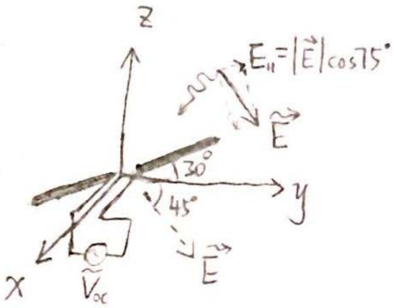
a) $+x$ direction

$$S_{av} = \frac{|\vec{E}|^2}{2\eta} = \frac{|E_y|^2 + |E_z|^2}{2\eta_0} = \frac{1+1}{2\times 377} = 0.00265\,(W/m^2)$$

b) ① $\tilde{V}_{oc} = \vec{E}\cdot\vec{l} = (\hat{y}-\hat{z})e^{-jx}\cdot\frac{\lambda}{2}(\frac{\sqrt{3}}{2}\hat{y}+\frac{1}{2}\hat{z}) = \frac{\sqrt{3}-1}{4}\lambda e^{-jx} \Rightarrow |\tilde{V}_{oc}| = \frac{\sqrt{3}-1}{4}\lambda.$

$k = 1 = \frac{2\pi}{\lambda} \Rightarrow \lambda = 2\pi\,(m).$ So $|\tilde{V}_{oc}| = \frac{\sqrt{3}-1}{2}\pi = 1.15\,(V).$

$$P_{int} = \frac{|\tilde{V}_{oc}|^2}{8R_{rad}} = \frac{1.15^2}{8\times 73} = 2.26\times10^{-3}\,(W).$$



② $P_{int} = S_{av}\cdot A_e\cdot\cos^2\theta_{polar}$

$S_{av} = 0.00265\,(W/m^2)$ as in (a)

$$A_e = \frac{\lambda^2 D}{4\pi} = \frac{(2\pi)^2\times 1.64}{4\pi} = 5.15\,(m^2)$$

$\theta_{polar} = 75°.$ So $P_{int} = 0.92\times10^{-3}\,(W).$

$$|\tilde{V}_{oc}| = \sqrt{8\cdot P_{int}\cdot R_{rad}} = \sqrt{8\times 0.92\times10^{-3}\times 73} = 0.73\,(V)$$

(Both sets of answers are acceptable, but in fact, ② is more accurate, because the assumption $\tilde{V}_{oc} = \vec{E}\cdot\vec{l}$ is not valid for $\lambda/2$ dipole (though it is approximately correct for short dipole). Remember the current distribution along the $\lambda/2$ dipole is a sinusoidal function (though still approximately), not as uniform as in the short dipole. So the voltage along the $\lambda/2$ dipole is a cosine function, not as linear as in the short dipole. Although the incident field is uniform, the induced voltage is still forced to follow a cosine function as a result of boundary conditions.)

c) Change the antenna axis to be along the same direction of $\vec{E}$, i.e. $(\frac{\sqrt{2}}{2}\hat{y}. -\frac{\sqrt{2}}{2}\hat{z})$. normalized

① $\tilde{V}_{oc} = (\hat{y}-\hat{z})e^{-jx} \cdot \frac{\lambda}{2}(\frac{\sqrt{2}}{2}\hat{y}-\frac{\sqrt{2}}{2}\hat{z}) = \frac{\sqrt{2}}{2}\lambda e^{-jx} \Rightarrow |\tilde{V}_{oc}| = \frac{\sqrt{2}}{2}\lambda = \sqrt{2}\pi = 4.44\ (V)$.

$P_{int} = \frac{|\tilde{V}_{oc}|^2}{8R_{rad}} = \frac{4.44^2}{8\times 73} = 0.0338\ (W)$

② $P_{int} = S_{av} \cdot A_e \cdot \cos^2 0 = 0.00265 \times 5.15 = 0.014\ (W)$

$|\tilde{V}_{oc}| = \sqrt{8 P_{int} R_{rad}} = \sqrt{8\times 0.014 \times 73} = 2.8\ (V)$

(Again, ② is more accurate.)

d) $l = 10mm = 0.01\,m$. $\frac{l}{\lambda} = \frac{0.01}{2\pi} = 0.0016 \ll 1$. It is a short dipole now.

① $\tilde{V}_{oc} = \vec{E}\cdot\vec{l} = (\hat{y}-\hat{z})e^{-jx} \cdot l\hat{z} = -le^{-jx}$

$\Rightarrow |\tilde{V}_{oc}| = 0.01\ (V)$

$R_{rad} = 80\pi^2 (l/\lambda)^2 = 80\pi^2 \times (0.0016)^2 = 2\times 10^{-3}\ (\Omega)$

So $P_{int} = \frac{|\tilde{V}_{oc}|^2}{8R_{rad}} = 6.25\times 10^{-3}\ (W)$

② $P_{int} = S_{av} \cdot A_e \cdot \cos^2(135°)$, $A_e = \frac{\lambda^2 D}{4\pi} = \frac{(2\pi)^2 \times 1.5}{4\pi} = 1.5\pi = 4.71\ (m^2)$.

So $P_{int} = 0.00265 \times 4.71 \times \cos^2(135°) = 6.25\times 10^{-3}\ (W)$

$|\tilde{V}_{oc}| = \sqrt{8 P_{int} R_{rad}} = \sqrt{8\times 6.25\times 10^{-3} \times 2\times 10^{-3}} = 0.01\ (V)$

(As now you can see, for short dipole, both methods lead to the same answer.)