# UCLA Department of Electrical Engineering
# EE101A – Engineering Electromagnetics
# Winter 2015
# Final Exam, March 17 2015, (3 hours)

Name ________________________________ Student number__________________________

This is a closed book exam – you are allowed 2 page of notes (each page front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit.

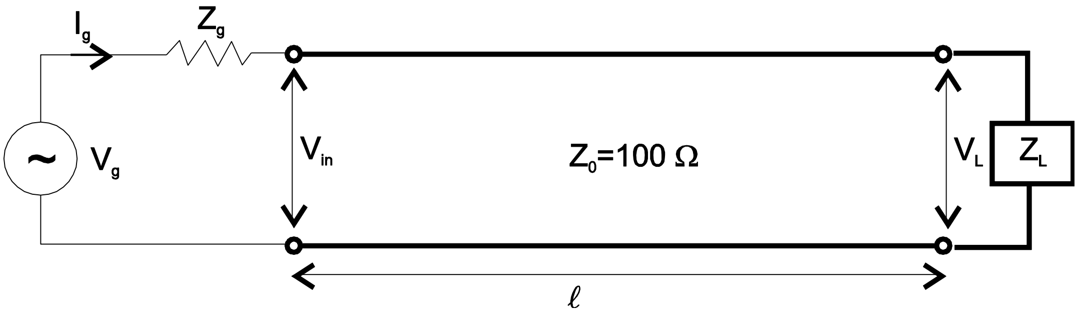
If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

| | Topic | Max Points | Your points |
|---|---|---|---|
| Problem 1 | Smith Chart | 15 | |
| Problem 2 | Impedance Matching | 40 | |
| Problem 3 | Impedance of Transmission Line | 15 | |
| Problem 4 | Phasors and Maxwell’s Eq | 30 | |
| | | | |
| Total | | 100 | |

**1. Smith chart basics (15 points)**

Consider the generic transmission line problem as shown below. Assume that the transmission line is a coaxial transmission line filled with a material that has $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$.




(a) (5 points) For each of the following loads, mark their position on the Smith chart below (using the letter as a label), and write below the reflection coefficient (magnitude and phase angle).

A: $Z_L$= 60 Ω. $\Gamma =$

B: $Z_L$ = 150 + $j$300 Ω. $\Gamma =$

(b) (5 points) For each of the following loads impedances, convert to unnormalized load admittance $Y_L$ and give the value in units $\Omega^{-1}$. Mark the position on the Smith chart below (using the letter as a label).

A: $Z_L$= 60 Ω. A' $Y_L$=

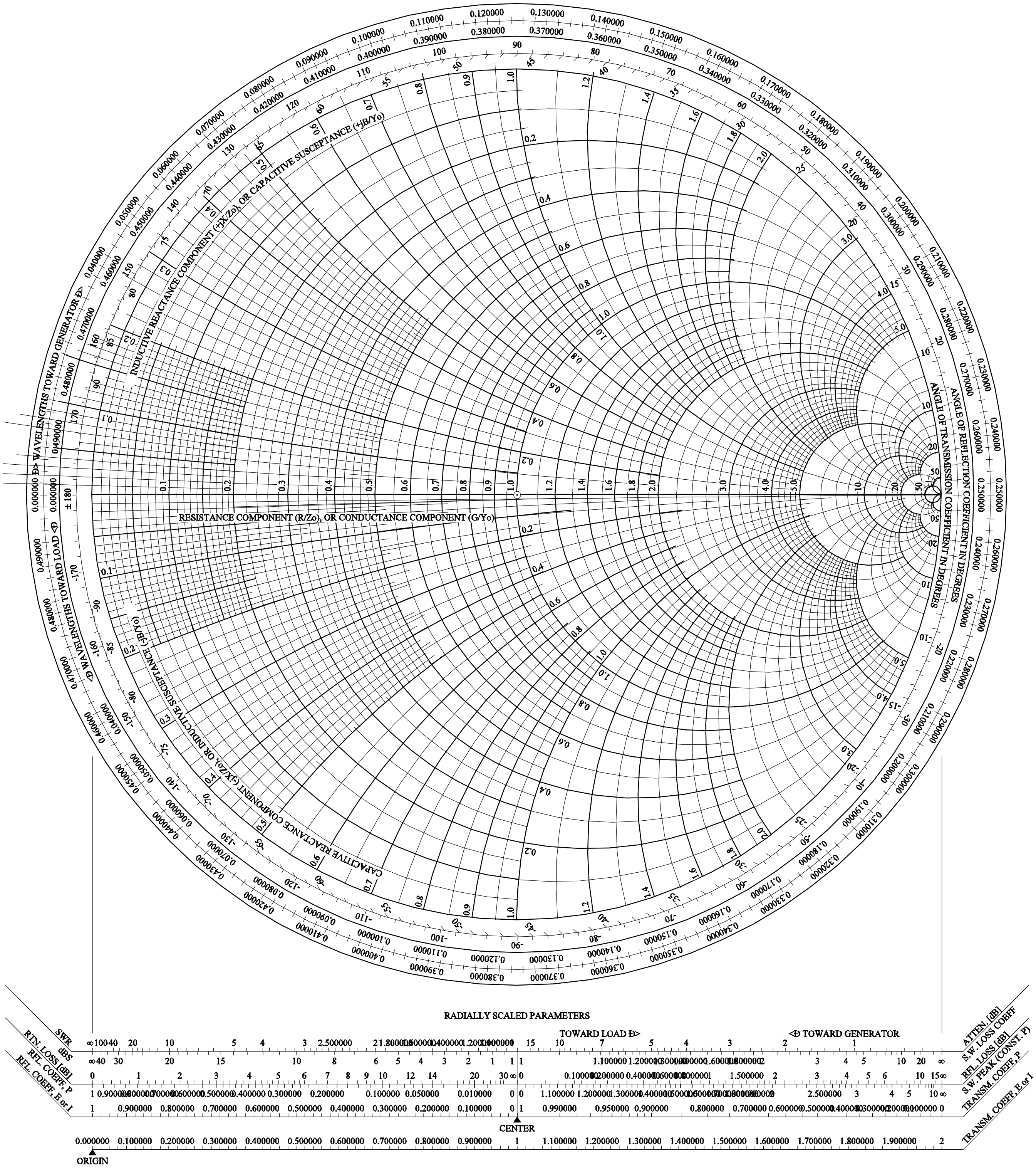
B: $Z_L$ = 150 + $j$300 Ω. B' $Y_L$=

(c) (5 points) What is the non-normalized input impedance of the transmission line $Z_{in}(-l)$ for each of the loads if $l$=2 cm and $f$=1 GHz? Label each point on the Smith Chart using A’’, B’’.

A: $Z_L$= 60 Ω. A’’ $Z_{in}$ =

B: $Z_L$ = 150 + $j$300 Ω. B’’ $Z_{in}$ =

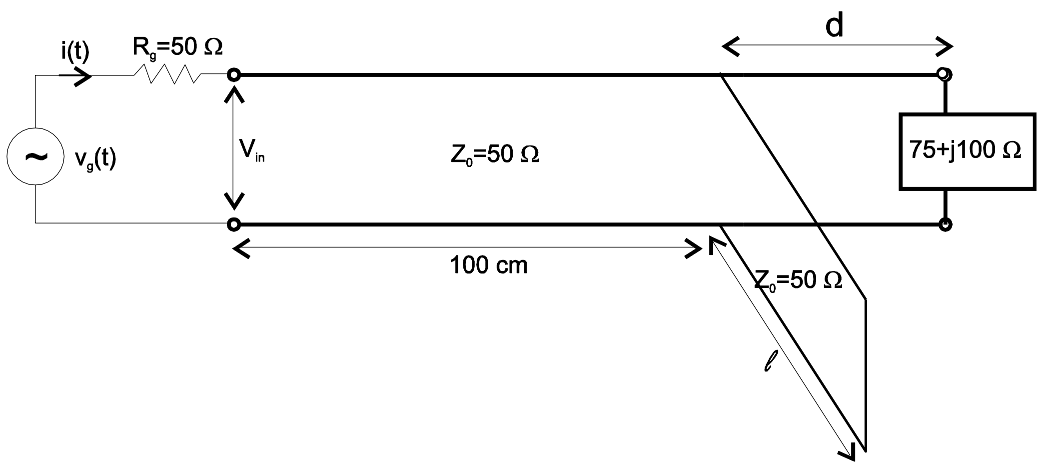
Smith chart for problem 1



2. **Transmission line – Impedance Matching** **(40 points)**

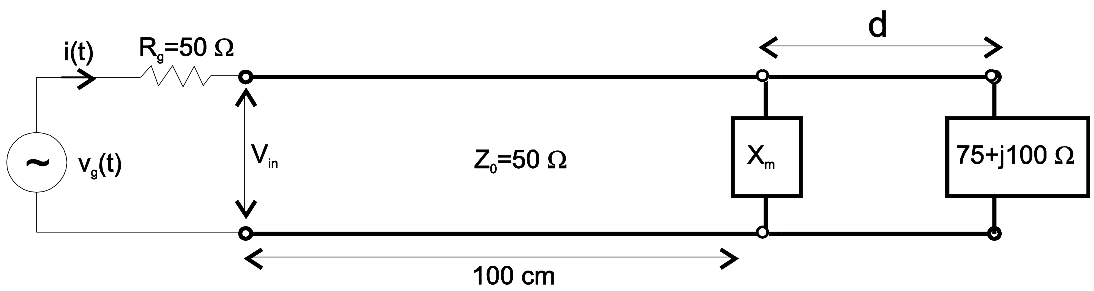
For this problem, you may use any methods you wish, including the Smith chart. Also, throughout this problem assume that the transmission line is coaxial filled with a dielectric material $\varepsilon=9\varepsilon_0$, $\mu=\mu_0$ , and the generator voltage is $v(t)=V_0\cos(2\pi f t)$, where $f$=5 GHz and $V_0$=1 V throughout the problem.

(a) (20 points) The goal of this problem is to design an impedance matching network that prevents any reflections into the network and maximizes the power delivered to the load, using a shorted stub. All transmission lines have the same characteristic impedance $Z_0$. Find the lengths $d$ and $\ell$ in order to impedance match the load the line. Give your answer in terms of wavelengths. (Note, there are multiple solutions – you only need give one).

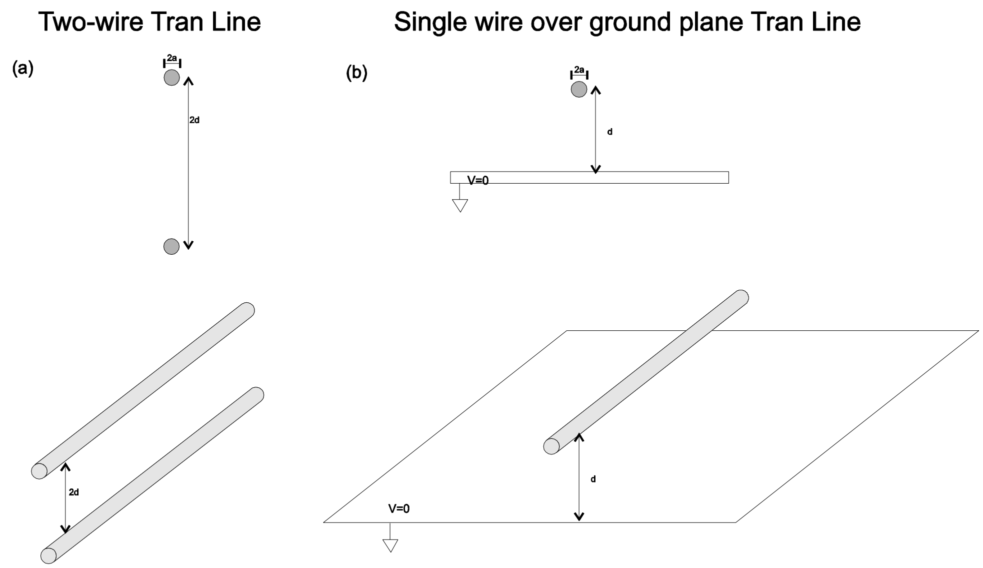


(b) (10 points) For the same problem, what should the length $d$ and $\ell$ be in meters?

(c) (10 points) Now consider using a lumped circuit element to match instead of a shorted stub (shown as reactance $X_m$ in the figure). Should you use an inductor or a capacitor? What value should you use (in either units Farads or Henries)?



3. **(15 points)** **Impedance of transmission line.** Consider a two wire transmission line, and a single wire transmission line over a ground plane with dimensions as shown (assume that *d* and *a* have the same values in each case). If the characteristic impedance of the two-wire line is $Z_0$=40 Ω, what is $Z_0$ for the wire over the ground plane? Explain the reasoning behind your answer in 1-3 sentences.



**4. Phasors and Maxwell's Equations (30 points)**

(a) (8 points) Write the following phasor quantities in the time domain assuming an angular frequency $\omega$. (Do not include the expression "Re{}" in your answer). Assume $E_0$, $H_0$, $V_0$, and $A$ are real numbers.

i. $\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_0 e^{-jkz}$ $\mathbf{E}(z,t) =$

ii. $\tilde{\mathbf{H}}(z) = \hat{\mathbf{y}} j H_0 e^{-jkz}$ $\mathbf{H}(z,t) =$

iii. $\tilde{F} = 3A(1-j)$ $F(t) =$

iv. $\tilde{V}(z) = V_0 \sin(\beta z)$ $V(z,t) =$

(b) (4 points) Consider a plane wave propagating through a particular medium with the phasor relations for the field:

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_0 e^{jkz}$$

$$\tilde{\mathbf{H}}(z) = -\hat{\mathbf{y}} \frac{E_0}{100} e^{jkz}$$

Assuming that $\mu = \mu_0$, what is the value of $\sigma$ and $\epsilon$? What direction is this wave propagating?

(c) (4 points)

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_0 e^{-\gamma z}$$

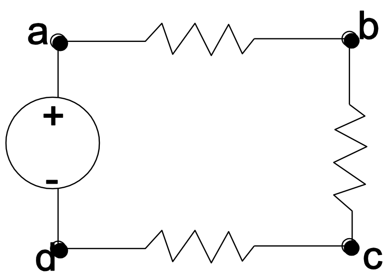
$$\tilde{\mathbf{H}}(z) = \hat{\mathbf{y}}(1-j)\frac{E_0}{8} e^{-\gamma z}$$

Assuming that $\mu = \mu_0$, is this wave propagating through good conductor or a poor conductor (i.e lossy dielectric)? Explain how you can tell the difference.

(d) (4 points) Consider the equation: $\nabla \bullet \mathbf{J} = -\frac{\partial \rho}{\partial t}$. Apply this equation to a volume $V$ with a surface defined by differential elements $d\mathbf{S}$, and rewrite this equation in integral form. Give a physical explanation of what conservation law that this describes.

(e) (5 points) In circuit theory, Kirchoff's voltage law says that the sum of voltages in a closed circuit must add up to zero. Qualitatively explain and/or derive how this rule can be derived from one of Maxwell's equations.

$$V_{ad}+V_{ba}+V_{cb}+V_{cd}=0$$



(f) (5 points)

Here is a "proof" that there is no such thing as magnetism. Magnetic Gauss's law states that: $\nabla \bullet \mathbf{B} = 0$,
When we apply the divergence theorem, we find:

$$\int_V (\nabla \bullet \mathbf{B})\, dV = \int_S \mathbf{B} \bullet d\mathbf{S} = 0.$$

Because **B** has zero divergence, we are able to define **B** as the curl of the vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$ If we combine the last two equations, we obtain:

$$\int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = 0.$$

Next we apply Stokes's theorem to the above result to obtain:

$$\int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = \oint_C \mathbf{A} \bullet d\mathbf{l} = 0$$

Thus we have shown that the circulation of **A** is path independent. It follows that we can write $\mathbf{A} = \nabla \psi$ where $\psi$ is some scalar function. Since the curl of a gradient is zero, we arrive at the remarkable conclusion that:

$$\mathbf{B} = \nabla \times (\nabla \psi) = 0.$$

That is, the magnetic field is zero everywhere!

Obviously I made a mistake somewhere in this proof. Explain where I went wrong.
(Hint: pay careful attention to the definitions of the various laws and theorems – it may be helpful to make sketches).

Maxwell's Equations:

$$\nabla \bullet \mathbf{D} = \rho_f$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \bullet \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Auxillary Fields:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In linear media: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$, $\mathbf{M} = \chi_m \mathbf{H}$  $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$

Ohm's Law: $\mathbf{J} = \sigma \mathbf{E}$  $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$

Electrostatic Potential: $\mathbf{E} = -\nabla V$  Vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$

Gradient Theorem: $\displaystyle\int_a^b (\nabla f) \bullet d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\displaystyle\int_V (\nabla \bullet \mathbf{A})\, dV = \oint_S \mathbf{A} \bullet d\mathbf{S}$

Stokes's Theorem: $\displaystyle\int_S (\nabla \times \mathbf{A}) \bullet d\mathbf{S} = \oint_C \mathbf{A} \bullet d\mathbf{l}$

Electric energy density: $W_e = \dfrac{1}{2} \mathbf{E} \bullet \mathbf{D}$ or $W_e = \dfrac{1}{2} \varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \dfrac{1}{2} \mathbf{B} \bullet \mathbf{H}$ or $W_m = \dfrac{1}{2} \mu H^2$ (in linear media)

Power dissipation density (Joule/Ohmic) = $\mathbf{E} \bullet \mathbf{J}$ or $\sigma E^2$ (in Ohm's law media)

Poynting Theorem: $\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\dfrac{\partial}{\partial t}\left(\dfrac{1}{2}\varepsilon E^2 + \dfrac{1}{2}\mu H^2\right) - \mathbf{E} \bullet \mathbf{J}$

Poynting Vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \dfrac{1}{2} \mathrm{Re}\left\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right\}$

Capacitance: $C = \dfrac{Q}{V}$

Inductance: $L = \dfrac{\Lambda}{I} = N \dfrac{\Phi}{I}$

Vector identities

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$$
$$\nabla \bullet (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times (\nabla f) = 0$$
$$\nabla \bullet \nabla f = \nabla^2 f$$
$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$$
$$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$$
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \bullet \mathbf{C}) - \mathbf{C}(\mathbf{A} \bullet \mathbf{B})$$

EM waves - Wave equation in source free medium, in time-domain and in harmonic (phasor) form

$$\nabla^2\mathbf{E} = \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} \qquad \nabla^2\mathbf{H} = \mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2} \qquad \nabla^2\tilde{\mathbf{E}} - \gamma^2\tilde{\mathbf{E}} = 0 \qquad \nabla^2\tilde{\mathbf{H}} - \gamma^2\tilde{\mathbf{H}} = 0$$

$$\gamma^2 = -\omega^2\mu\varepsilon \qquad \gamma = \alpha + j\beta \qquad k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} = \frac{n\omega}{c} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

z-propagating Plane wave (linearly polarized in x-direction) – phasor format – nonconducting media

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}E_{x0}^{+}e^{-jkz} \qquad \tilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{E_{x0}^{+}}{\eta}e^{-jkz}$$

Generalized plane wave in arbitrary direction with wavevector **k** with arbitrary linear polarization **e**.

$$\tilde{\mathbf{E}}(\mathbf{R}) = \hat{\mathbf{e}}E_0^{+}e^{-j\mathbf{k}\cdot\mathbf{R}} \qquad \tilde{\mathbf{E}} = -\eta\mathbf{k}\times\tilde{\mathbf{H}} \qquad \tilde{\mathbf{H}} = \frac{1}{\eta}\mathbf{k}\times\tilde{\mathbf{E}}$$

Conducting media

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \qquad \gamma = \alpha + j\beta \qquad \eta = \sqrt{\frac{\mu}{\varepsilon_c}} \qquad \delta_s = \frac{1}{\sqrt{\pi f\mu\sigma}} \qquad \tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}E_{x0}^{+}e^{-\gamma z} \qquad \tilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{E_{x0}^{+}}{\eta}e^{-\gamma z}$$

$$\alpha = \omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right]^{1/2} \qquad \beta = \omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right]^{1/2}$$

Transmission lines

$$-\frac{d\tilde{V}(z)}{dz} = \left(R' + j\omega L'\right)\tilde{I}(z) \qquad \frac{d^2\tilde{V}(z)}{dz^2} - \gamma\tilde{V}(z) = 0 \qquad \gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$$-\frac{d\tilde{I}(z)}{dz} = \left(G' + j\omega C'\right)\tilde{V}(z) \qquad \frac{d^2\tilde{I}(z)}{dz^2} - \gamma\tilde{I}(z) = 0 \qquad Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}}$$

Lossless transmission lines:

$$\gamma = j\beta = j\omega\sqrt{L'C'} \qquad Z_0 = \sqrt{\frac{L'}{C'}} \qquad u_p = \frac{1}{\sqrt{L'C'}} \qquad \beta = \frac{2\pi}{\lambda} = \frac{\omega}{u_p}$$

TEM lossless transmission lines:

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon} \qquad Z_0 = \sqrt{\frac{L'}{C'}} \qquad u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

Transmission line wave solutions (lossless lines)

$$\tilde{V}(z) = V_0^{+}e^{-j\beta z} + V_0^{-}e^{j\beta z} \qquad \tilde{I}(z) = \frac{V_0^{+}}{Z_0}e^{-j\beta z} - \frac{V_0^{-}}{Z_0}e^{j\beta z}$$

$$\Gamma = \frac{V_0^{-}}{V_0^{+}} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad VSWR = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0\frac{1+\Gamma e^{j2\beta z}}{1-\Gamma e^{j2\beta z}} \qquad Z_{in}(z=-l) = Z_0\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}$$

Impedance: $Z = R + jX = \frac{1}{Y} = \frac{1}{G + jB}$ Admittance: $Y = G + jB$

Constants (SI units): $\varepsilon_0$=8.85x10$^{-12}$ F/m (or C$^2$ N$^{-1}$ m$^{-2}$) $\mu_0$=4π x10$^{-7}$ H/m (or N A$^{-2}$)

**Table 3-1:** Summary of vector relations.

| | Cartesian Coordinates | Cylindrical Coordinates | Spherical Coordinates |
|---|---|---|---|
| **Coordinate variables** | $x, y, z$ | $r, \phi, z$ | $R, \theta, \phi$ |
| **Vector representation, A =** | $\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ | $\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ | $\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$ |
| **Magnitude of A, $\lvert A \rvert$ =** | $\sqrt{A_x^2 + A_y^2 + A_z^2}$ | $\sqrt{A_r^2 + A_\phi^2 + A_z^2}$ | $\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$ |
| **Position vector $\overrightarrow{OP_1}$ =** | $\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$ | $\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$ | $\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$ |
| **Base vectors properties** | $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$<br>$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$<br>$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$<br>$\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$<br>$\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$ | $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$<br>$\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$<br>$\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$<br>$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$<br>$\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$ | $\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$<br>$\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$<br>$\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$<br>$\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$<br>$\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$ |
| **Dot product, A · B =** | $A_xB_x + A_yB_y + A_zB_z$ | $A_rB_r + A_\phi B_\phi + A_zB_z$ | $A_RB_R + A_\theta B_\theta + A_\phi B_\phi$ |
| **Cross product, A × B =** | $\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$ | $\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$ | $\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$ |
| **Differential length, $d\mathbf{l}$ =** | $\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$ | $\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$ | $\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$ |
| **Differential surface areas** | $d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$<br>$d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$<br>$d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$ | $d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$<br>$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$<br>$d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$ | $d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$<br>$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$<br>$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$ |
| **Differential volume, $d\mathcal{V}$ =** | $dx\,dy\,dz$ | $r\,dr\,d\phi\,dz$ | $R^2\sin\theta\,dR\,d\theta\,d\phi$ |

**Table 3-2:** Coordinate transformation relations.

| Transformation | Coordinate Variables | Unit Vectors | Vector Components |
|---|---|---|---|
| **Cartesian to cylindrical** | $r = \sqrt{x^2 + y^2}$<br>$\phi = \tan^{-1}(y/x)$<br>$z = z$ | $\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$<br>$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{z}}$ | $A_r = A_x\cos\phi + A_y\sin\phi$<br>$A_\phi = -A_x\sin\phi + A_y\cos\phi$<br>$A_z = A_z$ |
| **Cylindrical to Cartesian** | $x = r\cos\phi$<br>$y = r\sin\phi$<br>$z = z$ | $\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$<br>$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{z}}$ | $A_x = A_r\cos\phi - A_\phi\sin\phi$<br>$A_y = A_r\sin\phi + A_\phi\cos\phi$<br>$A_z = A_z$ |
| **Cartesian to spherical** | $R = \sqrt{x^2 + y^2 + z^2}$<br>$\theta = \tan^{-1}[\sqrt{x^2 + y^2}/z]$<br>$\phi = \tan^{-1}(y/x)$ | $\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi$ $+ \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$<br>$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi$ $+ \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$<br>$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ | $A_R = A_x\sin\theta\cos\phi$ $+ A_y\sin\theta\sin\phi + A_z\cos\theta$<br>$A_\theta = A_x\cos\theta\cos\phi$ $+ A_y\cos\theta\sin\phi - A_z\sin\theta$<br>$A_\phi = -A_x\sin\phi + A_y\cos\phi$ |
| **Spherical to Cartesian** | $x = R\sin\theta\cos\phi$<br>$y = R\sin\theta\sin\phi$<br>$z = R\cos\theta$ | $\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$<br>$\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi$ $+ \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$<br>$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ | $A_x = A_R\sin\theta\cos\phi$ $+ A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$<br>$A_y = A_R\sin\theta\sin\phi$ $+ A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$<br>$A_z = A_R\cos\theta - A_\theta\sin\theta$ |
| **Cylindrical to spherical** | $R = \sqrt{r^2 + z^2}$<br>$\theta = \tan^{-1}(r/z)$<br>$\phi = \phi$ | $\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$<br>$\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$<br>$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ | $A_R = A_r\sin\theta + A_z\cos\theta$<br>$A_\theta = A_r\cos\theta - A_z\sin\theta$<br>$A_\phi = A_\phi$ |
| **Spherical to cylindrical** | $r = R\sin\theta$<br>$\phi = \phi$<br>$z = R\cos\theta$ | $\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$<br>$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$<br>$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ | $A_r = A_R\sin\theta + A_\theta\cos\theta$<br>$A_\phi = A_\phi$<br>$A_z = A_R\cos\theta - A_\theta\sin\theta$ |

## GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

### CARTESIAN (RECTANGULAR) COORDINATES $(x, y, z)$

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

### CYLINDRICAL COORDINATES $(r, \phi, z)$

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

### SPHERICAL COORDINATES $(R, \theta, \phi)$

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

## SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin \theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(U + V) = \nabla U + \nabla V$$

$$\nabla(UV) = U \nabla V + V \nabla U$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \cdot (U\mathbf{A}) = U \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$$

$$\nabla \times (U\mathbf{A}) = U \nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

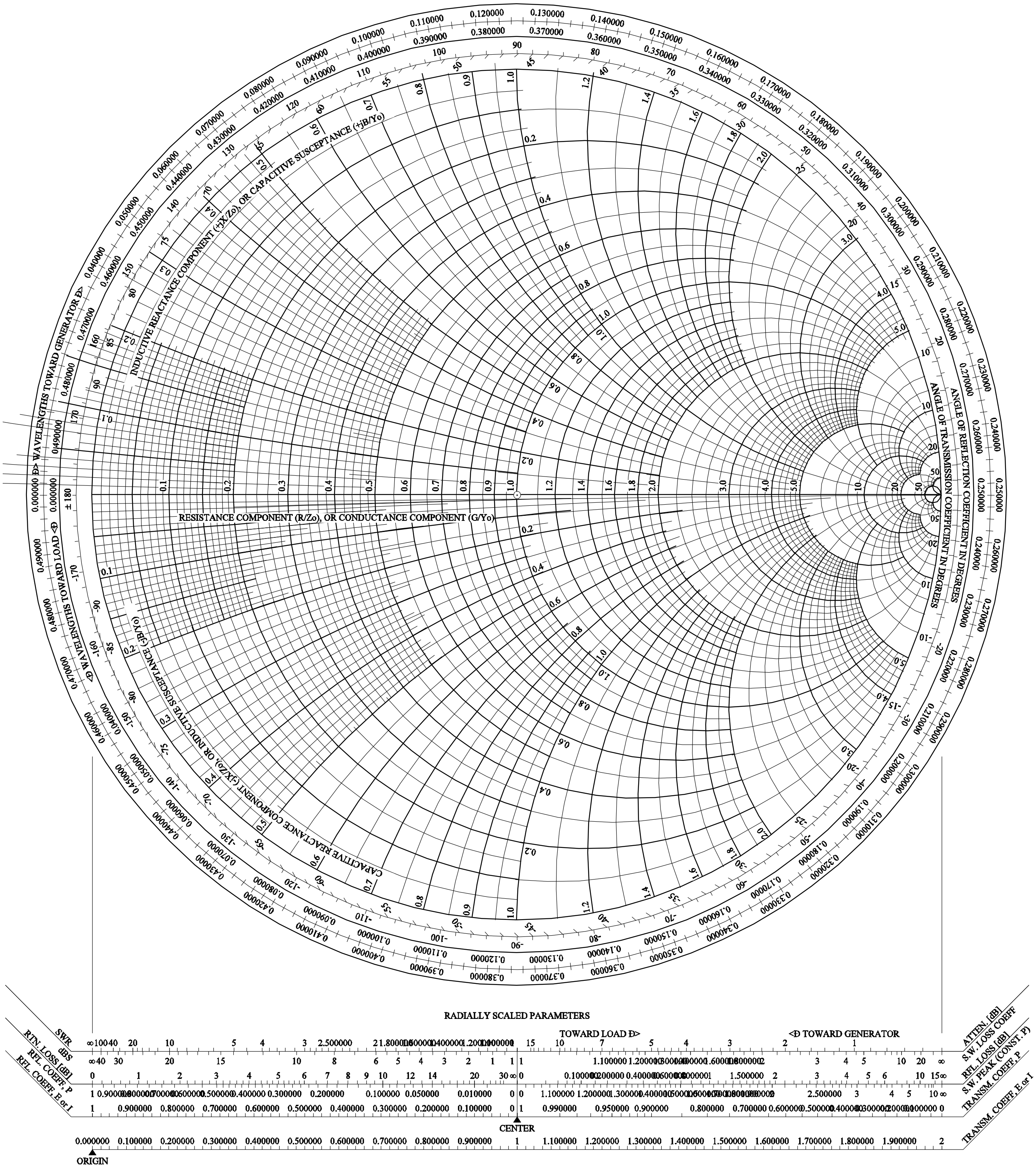
$$\nabla \times \nabla V = 0$$

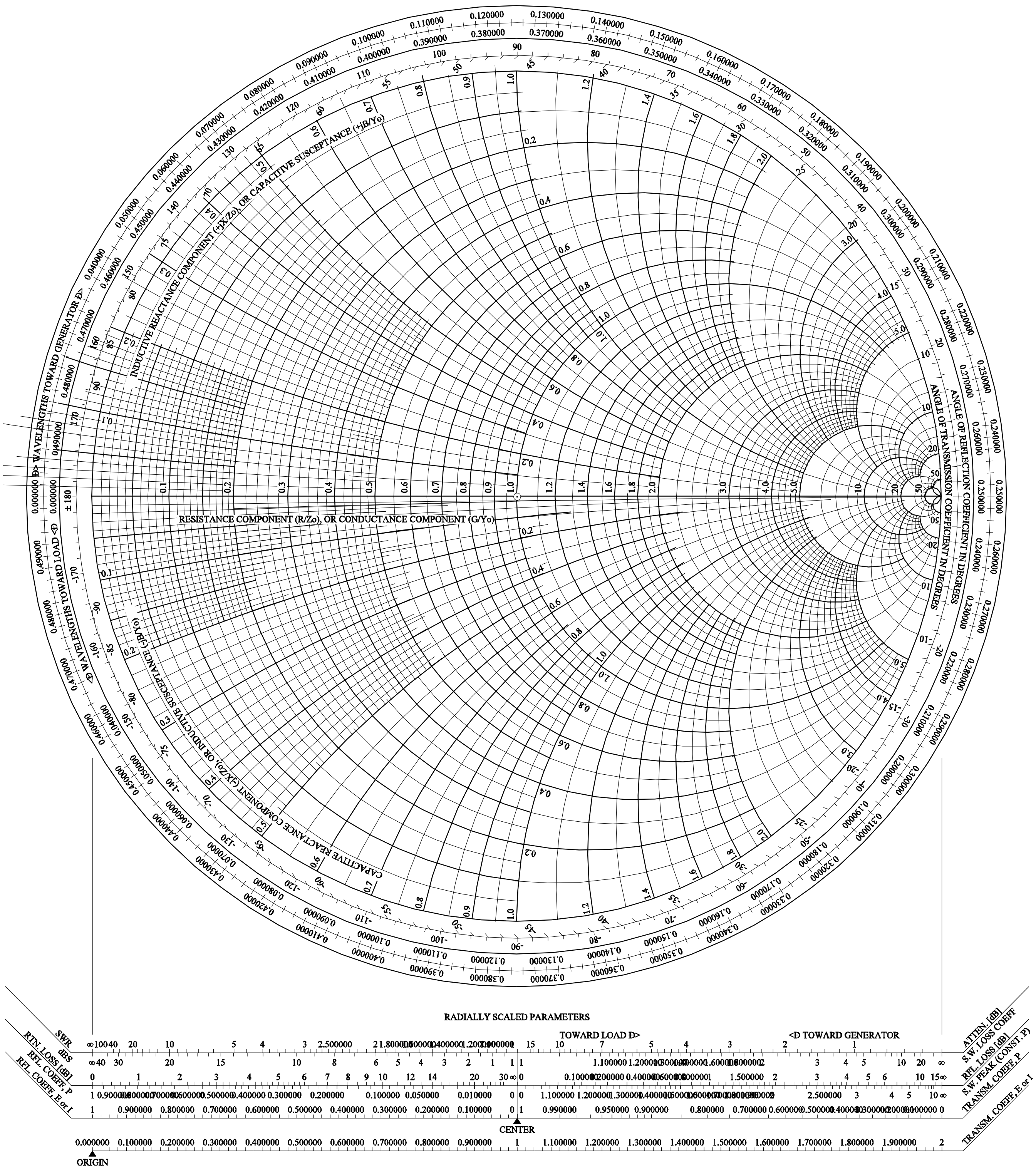
$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$\displaystyle\int_{\mathcal{V}} (\nabla \cdot \mathbf{A})\, d\mathcal{V} = \oint_S \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem ($S$ encloses $\mathcal{V}$)

$\displaystyle\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem ($S$ bounded by $C$)


WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
ANGLE OF TRANSMISSION COEFFICIENT IN DEGREES
RADIALLY SCALED PARAMETERS
TOWARD LOAD
TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER
ORIGIN



RESISTANCE COMPONENT (R/Zo), OR CONDUCTANCE COMPONENT (G/Yo)
INDUCTIVE REACTANCE COMPONENT (+jX/Zo), OR CAPACITIVE SUSCEPTANCE (+jB/Yo)
CAPACITIVE REACTANCE COMPONENT (-jX/Zo), OR INDUCTIVE SUSCEPTANCE (-jB/Yo)
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TOWARD LOAD
TOWARD GENERATOR
SWR
dBS
RTN. LOSS [dB]
RFL. COEFF, P
RFL. COEFF, E or I
ATTEN. [dB]
S.W. LOSS COEFF
RFL. LOSS [dB]
S.W. PEAK (CONST. P)
TRANSM. COEFF, P
TRANSM. COEFF, E or I
CENTER
ORIGIN

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