SOLUTIONS

UCLA Department of Electrical Engineering
EE101A – Engineering Electromagnetics
Winter 2016
Midterm, February 8 2015, (1:45 minutes)

Name ______________________________ Student number ______________________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	40	
Problem 2	Conductors and fields	30	
Problem 3	Inductance	30	
Total		100	

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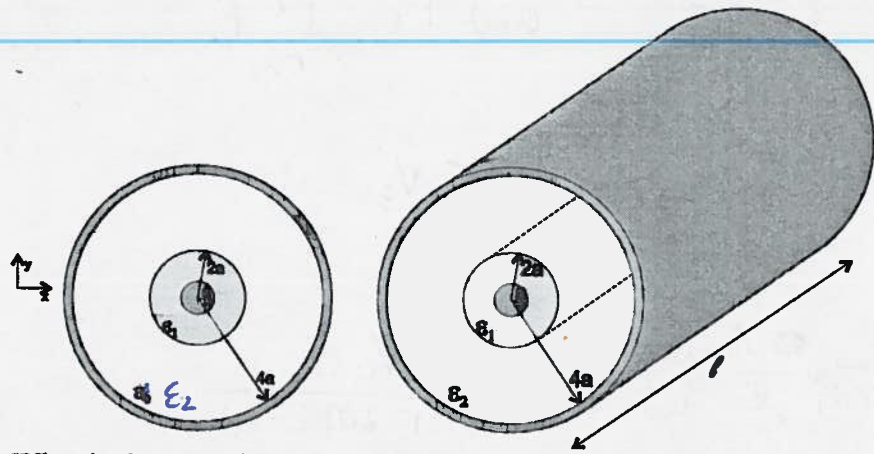
1. Coaxial capacitor (40 points)

Consider a piece of coaxial cable of length ℓ with two dielectric layers with permittivities ε_1 and ε_2. You may consider the inner conductor (radius a) and the outer conductor shell (radius $4a$) to be perfect conductors.



(a) (20 points) What is the capacitance C between the inner and outer conductors? (Your answer should be in terms of only geometric and material parameters such as ℓ, a, ε_1, ε_2)

Find E-field assuming line charge density ρ_ℓ.

$$\vec{D} = \frac{\rho_\ell}{2\pi r}\hat{r} \qquad \vec{E}_1 = \hat{r}\frac{\rho_\ell}{2\pi r\varepsilon_1}\ (a<r<2a) \qquad \vec{E}_2 = \hat{r}\frac{\rho_\ell}{2\pi r\varepsilon_2}\ (2a<r<4a)$$

$$V(a)-V(4a) = V(a)-V(2a)+V(2a)-V(4a) \qquad \text{Let } V(4a)=0$$

$$V(2a) = -\int_{4a}^{2a}\frac{\rho_\ell}{2\pi\varepsilon_2 r}dr = -\frac{\rho_\ell}{2\pi\varepsilon_2}(\ln 2a - \ln 4a) = \frac{\rho_\ell}{2\pi\varepsilon_2}\ln 2$$

$$V(a)-V(2a) = -\int_{2a}^{a}\frac{\rho_\ell}{2\pi\varepsilon_1 r}dr = \frac{-\rho_\ell}{2\pi\varepsilon_1}(\ln a - \ln 2a) = \frac{\rho_\ell}{2\pi\varepsilon_1}\ln 2$$

$$V_0 = V(a)-V(4a) = \frac{\rho_\ell \ln 2}{2\pi}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)$$

$$C = \frac{\rho_\ell \ell}{V_0} = \frac{\rho_\ell \ell\, 2\pi}{\rho_\ell \ln 2\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)} = \frac{\ell\, 2\pi\, \varepsilon_1\varepsilon_2}{\ln(2)\,(\varepsilon_1+\varepsilon_2)}$$

(b) (20 point) If a potential difference of V_0 volts is applied between the inner and outer conductor such that $V(r = a)$-$V(r = 4a) = V_0$ Volts, write an expression for the electric field inside the dielectric (i.e. between $r=a$ and $r=4a$) in terms of V_0. There should not be an expression for charge or charge density in your final expression.

Since $E_1 = \frac{\rho_\ell}{2\pi r \varepsilon_1}\hat{r}$ and $E_2 = \frac{\rho_\ell}{2\pi r \varepsilon_2}\hat{r}$

and $C = \frac{\rho_\ell l}{V_0}$ so $\rho_\ell = \frac{C V_0}{l}$

$$E_1 = \frac{\hat{r}}{2\pi r \varepsilon_1} \frac{V_0}{l} \frac{l\, 2\pi\, \varepsilon_1 \varepsilon_2}{\ln 2\,(\varepsilon_1+\varepsilon_2)} = \frac{\hat{r}\, V_0\, \varepsilon_2}{r \ln(2)(\varepsilon_1+\varepsilon_2)}$$

so

$$E = \begin{cases} \hat{r}\,\dfrac{V_0\, \varepsilon_2}{r \ln(2)(\varepsilon_1+\varepsilon_2)} & a < r < 2a \\ \hat{r}\,\dfrac{V_0\, \varepsilon_1}{r \ln(2)(\varepsilon_1+\varepsilon_2)} & 2a < r < 4a \end{cases}$$

2. Conductors and fields (30 points)

(a) (10 points) Explain qualitatively/physically why the E-field must go to zero inside a perfect conductor.

Any applied E-field will cause current to flow. This will continue until the charge has arranged itself in such a way to cancel any applied fields. This takes approximately several dielectric relaxation times ($\tau_e = \varepsilon/\sigma$) to occur. Any net charge must reside on surface so that $\rho = 0$ inside conductor and $\nabla \cdot E = 0$ inside.

(b) (10 points) Explain qualitatively/physically why the B-field must go to zero inside a perfect conductor.

Any time changing B-field flux extending through a conductor will cause solenoidal E-fields (according to $\nabla \times E = -\frac{\partial B}{\partial t}$) and hence solenoidal currents (according to $J = \sigma E$). These currents flow in such a way to produce a B-field that cancels out the changing applied field (according to $\nabla \times B = \mu J$).

(c) (10 points) In a real conductor where σ is finite, we can only make approximations. Does a finite conductor act more like a perfect conductor at high or low frequencies for E-fields? How about for B-fields? Explain why.

A finite conductor acts as a perfect conductor at long times / low frequencies for $\vec{E}$. When $t \gg \tau_e = \frac{\varepsilon}{\sigma}$ the material has sufficient time for the charges to move to cancel the fields.

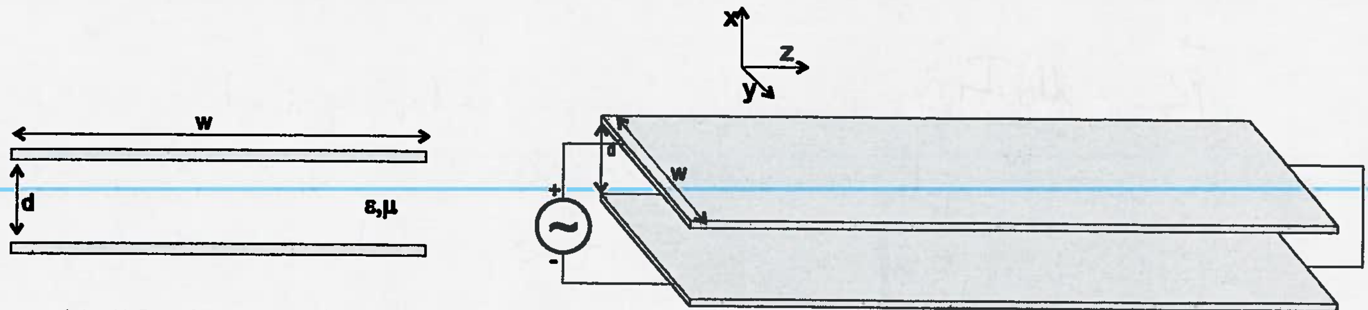
A finite conductor acts as a perfect conductor for $\vec{B}$ at short times / high frequencies. Any currents created by a changing B-field have an impulse response with time constant τ_m (i.e. the magnetic diffusion time).

For $t \ll \tau_m$ or $\omega \gg \frac{1}{\tau_m}$, induced currents effectively screen B-fields, because the currents haven't had time to die out.

Alternatively, the amount of driving current is $J \propto \nabla \times E \propto j\omega B$

which is larger at larger frequencies.

3. Inductance of parallel plate transmission lines (30 points)

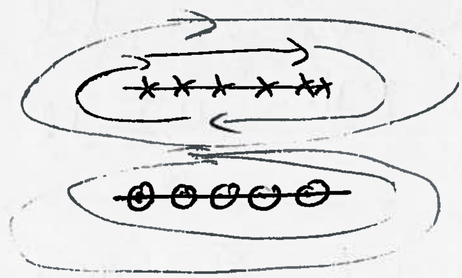


(a) (15 points) Consider the parallel plate transmission line shown above composed of two perfectly conducting thin plates. The width is much greater than the plate separation $w \gg d$. Assume that current $+I$ is flowing on the top plate (i.e. in the z direction), and current $-I$ is flowing on the bottom plate (i.e. in the $-z$ direction) (as is implied by the circuit diagram above). Give an expression for the B-field in between the plates. Your answer should be in terms of I, and the geometric and material parameters (make sure to include the vector direction).

Top plate : $\vec{J}_s = I/w \, \hat{z}$

Bottom plate : $\vec{J}_s = -I/w \, \hat{z}$

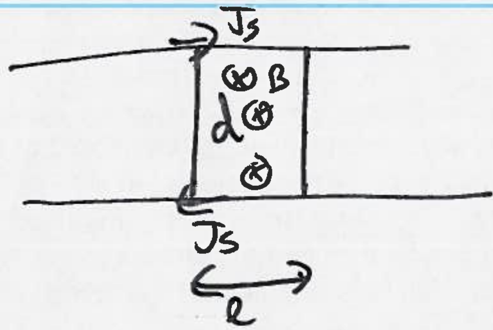
$\vec{H} = -\frac{I}{w}\hat{y}$ in between plates



$\vec{B} = -\mu_0 \frac{I}{w}\hat{y}$ in between plates

(b) (15 points) What is the inductance per unit length for this parallel plate transmission line?

$$\vec{B} = -\frac{\mu_0 I}{W}\hat{y}$$



Flux through square area $d \times l$

$$\Phi = Bdl = \frac{\mu_0 I}{W} dl$$

$$\frac{L}{l} = \frac{\Phi}{Il} = \frac{\mu_0 d}{w}$$

Maxwell's Equations in media:

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Auxillary Fields: $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$

In linear media: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ $\quad \mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{M} = \chi_m \mathbf{H}$ $\quad \mathbf{B} = \mu \mathbf{H}$

Ohm's law: $\mathbf{J}_f = \sigma \mathbf{E}$

Electrostatic Scalar Potential: $\mathbf{E} = -\nabla V$ $\quad$ Vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla \cdot \mathbf{A})\, dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$

Electric energy density: $W_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$ or $W_e = \frac{1}{2} \varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \frac{1}{2} \mathbf{B} \cdot \mathbf{H}$ or $W_m = \frac{1}{2} \mu H^2$ (in linear media)

Joule power dissipation density: $W_P = \mathbf{E} \cdot \mathbf{J}$ or $W_m = \sigma E^2$ (in Ohm's law media)

Poynting Vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \frac{1}{2} \mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}$

Capacitance: $C = \frac{Q}{V}$

Inductance: $L = \frac{\Lambda}{I} = N \frac{\Phi}{I}$

Boundary conditions: $E_{t,2} - E_{t,1} = 0$ $\quad H_{t,1} - H_{t,2} = J_s$

$D_{n,2} - D_{n,1} = \rho_s$ $\quad B_{n,2} - B_{n,1} = 0$

Bound charge: $\rho_{b,v} = -\nabla \cdot \mathbf{P}$ $\quad \rho_{b,s} = \mathbf{P} \cdot \hat{\mathbf{n}}$

Bound current: $\mathbf{J}_{b,v} = \nabla \times \mathbf{M}$ $\quad \mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}}$

Definition of phasor $\tilde{F}$ for time harmonic function $f(t)$:

$$\begin{cases} f(t) = \mathrm{Re}\{\tilde{F} e^{j\omega t}\} = |F| \cos(\omega t + \phi) \\ \tan^{-1}(\phi) = \mathrm{Im}\{\tilde{F}\} / \mathrm{Re}\{\tilde{F}\} \end{cases}$$

Constants (SI units): $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m (or $C^2\, N^{-1}\, m^{-2}$) $\quad \mu_0 = 4\pi \times 10^{-7}$ H/m (or $N\, A^{-2}$)

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A \rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

GRADIENT, DIVERGENCE, CURL & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing **A** and **B**

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(U + V) = \nabla U + \nabla V$$

$$\nabla(UV) = U\nabla V + V\nabla U$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$$

$$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times \nabla V = 0$$

$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$\int_{v} (\nabla \cdot \mathbf{A}) dv = \oint_{S} \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses v)

$\int_{S} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{C} \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)

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