SOLUTIONS

UCLA Department of Electrical Engineering
EE101 – Engineering Electromagnetics
Winter 2013
Midterm, February 12 2013, (1:45 minutes)

Name ______________________ Student number______________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	40	
Problem 2	Transmission Line	30	
Problem 3	Boundary condition	30	
Total		100	

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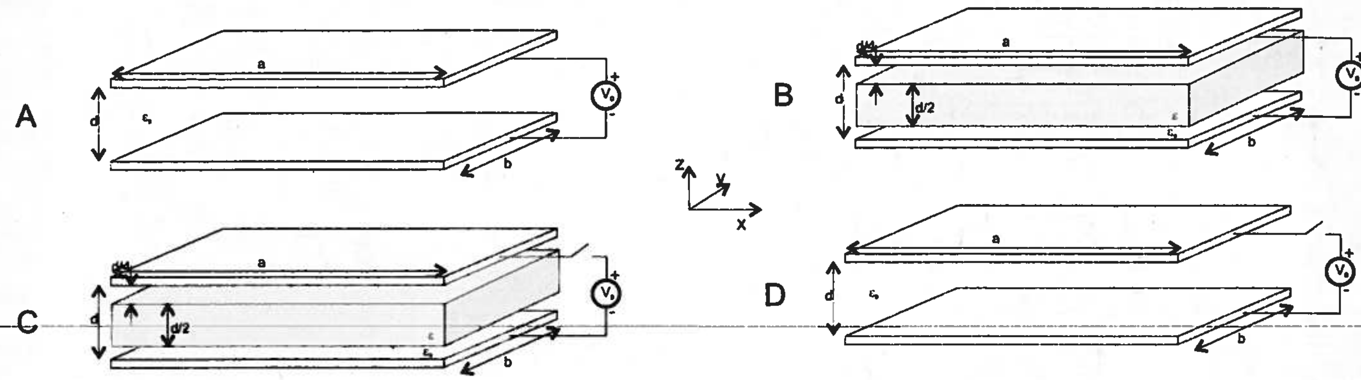
1. Capacitor (40 points)



(a) (10 points) Consider the following parallel plate capacitor with perfectly conducting metal plates, and only vacuum in between as shown below in figure (A). Assume the plates are held at a constant potential difference V_0 using a voltage source. Give an expression for the electric field in the gap between the plates (don't forget vector direction) in terms of V_0, and the dimensional and material quantities (i.e. a, b, d, ε, - (NOT C!)).

$$\boxed{\vec{E} = -\hat{z}\frac{V_0}{d}}$$

(b) (15 points) Now, we insert a piece of dielectric of thickness $d/2$ and with permittivity ε halfway in between the plates as shown in figure (B). Give an expression for electric field E (direction and magnitude) both inside the dielectric and in the vacuum regions.

Field in air: $\vec{E}_2 = -\hat{z} E_2$

Field in dielectric $\vec{E}_1 = -\hat{z} E_1$

Bnd cond.

$$\varepsilon_0 E_2 = \varepsilon E_1$$

$$E_2 = \frac{\varepsilon}{\varepsilon_0} E_1$$

$$\frac{d}{2} E_2 + \frac{d}{2} E_1 = V_0$$

$$\frac{E_1 + E_2}{2} = \frac{V_0}{d}$$

$$E_1 \left(\frac{\frac{\varepsilon}{\varepsilon_0} + 1}{2} \right) = \frac{V_0}{d}$$

$$\boxed{E_1 = \frac{2V_0}{d\left(1 + \frac{\varepsilon}{\varepsilon_0}\right)} = \frac{2V_0 \varepsilon_0}{d(\varepsilon_0 + \varepsilon)}} \quad \vec{E}_1 = -\hat{z} E_1$$

$$\boxed{E_2 = \frac{\varepsilon}{\varepsilon_0} E_1 = \frac{2V_0 \varepsilon}{d(\varepsilon_0 + \varepsilon)} \quad \vec{E}_2 = -\hat{z} E_2}$$

$\frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$

$C = \frac{Q}{V}$ $V = \frac{Q}{C}$

(c) (15 points) Now imagine that we open a switch connecting the voltage source (as shown in (C)) and remove the dielectric (as shown in (D)). Is the electrostatic energy in the system the same, larger, or smaller than the original configuration shown in (A)? If your answer is "same", explain why. If your answer is "larger" or "smaller", explain why, and where the energy came from or went to. (Explanations required for full credit)

When the switch is opened, the charge $\pm Q$ on each plate is fixed at its previous value.

$$\rho_s = \pm \varepsilon_0 E_z = \pm \frac{2V_0 \varepsilon \varepsilon_0}{d(\varepsilon_0 + \varepsilon)} \Rightarrow Q = \pm ab \frac{2V_0 \varepsilon \varepsilon_0}{d(\varepsilon_0 + \varepsilon)}$$

When ~~plate is~~ dielectric is removed, the E-field is constant at the value $\vec{E}_{new} = -\hat{z}\frac{\rho_s}{\varepsilon_0} = \frac{2V_0\varepsilon}{d(\varepsilon_0+\varepsilon)}$

Electrostatic energy density in (A): $W_e = \left(\frac{1}{2}\varepsilon E^2\right) \times abd$

$$W_e = \frac{1}{2}\varepsilon_0 \frac{V_0^2}{d^2} abd = \frac{\varepsilon_0 V_0^2 ab}{2d}$$

Electrostatic energy in (D) $W_e = \left(\frac{1}{2}\varepsilon E_{new}^2\right) abd$

$$= \frac{1}{2}\frac{\varepsilon_0 4 V_0^2 \varepsilon^2}{d^2(\varepsilon_0+\varepsilon)^2} abd$$

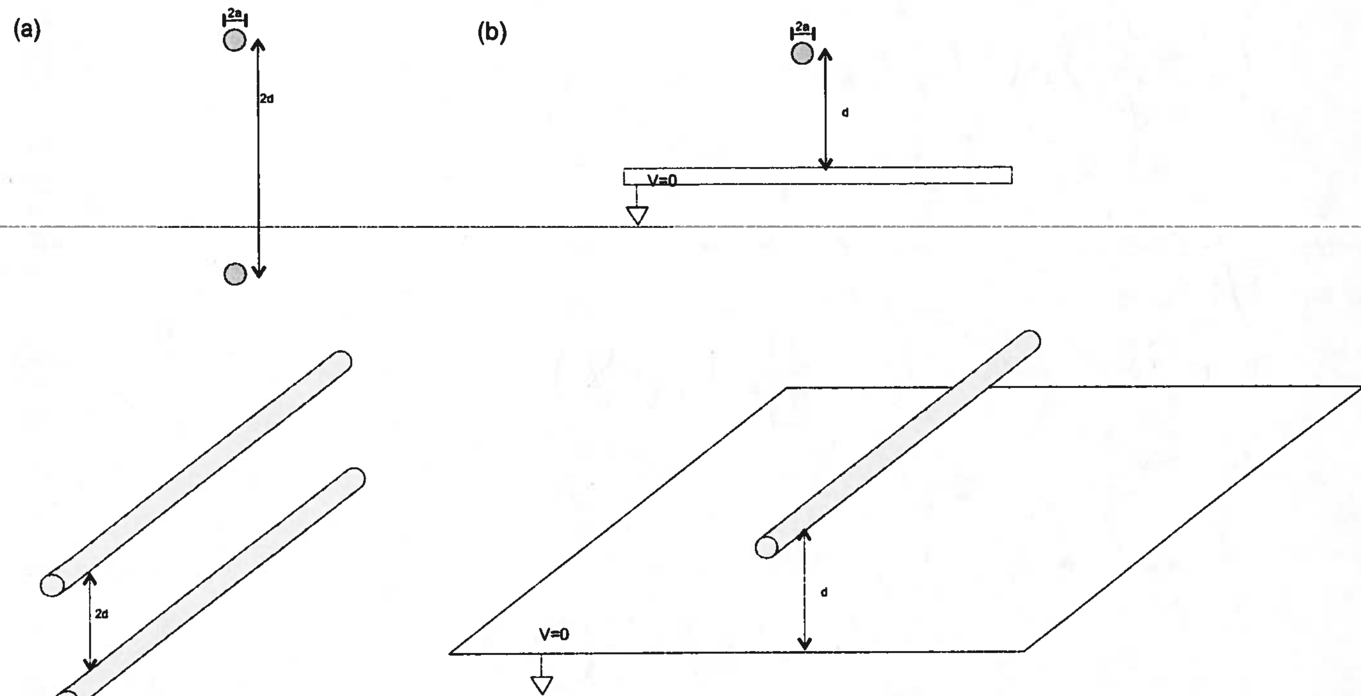
$$W_e = \frac{\varepsilon_0 V_0^2 ab}{2d\left(\frac{\varepsilon_0+\varepsilon}{2\varepsilon}\right)^2} = \frac{\varepsilon_0 V_0^2 ab}{2d\left(\frac{1+\varepsilon_0/\varepsilon}{2}\right)}$$

A: Electrostatic energy is larger in case D. When the switch is opened, the charge $\pm Q$ on the plates is fixed at a value larger than in case A. This results in a larger E-field intensity inside the gap, & larger electrostatic energy stored in field. The energy comes from the work done to remove the dielectric slab from the capacitor.

2. Transmission line (30 points)

Consider the lossless two-wire transmission line in case (a) and the single wire over a perfectly conducting semi-infinite ground plane shown in case (b)?

Two-wire Tran Line **Single wire over ground plane Tran Line**



(a) (15 points) The capacitance per unit length of the two-wire line is approximately: $\boxed{C' = \frac{C}{\ell} = \frac{\pi\varepsilon}{\ln(d/2a)}}$. What is the capacitance per unit length of the single wire over ground plane?

typo

$$C' = \frac{\pi\varepsilon}{\ln(2d/a)}$$

Correct Answer

$$C' = \frac{2\pi\varepsilon}{\ln(2d/a)}$$

Because of typo we also accept original answer $C' = \frac{2\pi\varepsilon}{\ln(d/2a)}$

(b) (15 points) The inductance per unit length of the two-wire line is approximately:

$L' = \frac{L}{\ell} = \frac{\mu}{\pi}\ln(d/2a)$. What is the inductance per unit length of the single wire over ground plane?

$\downarrow$ typo

$L' = \frac{\mu}{\pi} \ln(2d/a)$ correct

Correct Answer

Wire over plane: $L' = \frac{\mu}{2\pi} \ln(2d/a)$

Because of typo we also accept

$L' = \frac{\mu}{2\pi} \ln(2d/2a)$

3. Boundary conditions (30 points)

Consider the case of time-harmonic fields described by phasors $\tilde{H}$, $\tilde{B}$ for (i.e. fields varying with a single angular frequency ω).

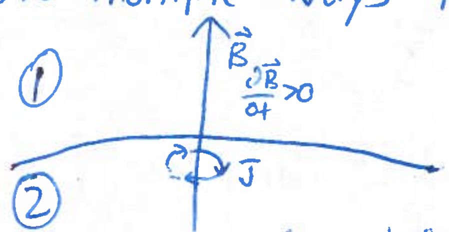
(a) (15 points) At the interface between a perfect conductor (region 2) and a material with permittivity ε and permeability μ (region 1), we can write the boundary conditions for magnetic fields:

$$\begin{cases} \tilde{H}_{1t} = \tilde{J}_s, \tilde{H}_{2t} = 0 & \text{(transverse fields)} \\ \tilde{B}_{1n} = \tilde{B}_{2n} = 0 & \text{(normal fields)} \end{cases}$$

$\vec{B}(t) = Re\{\tilde{\vec{B}} e^{j\omega t}\}$

Explain qualitatively why the normal B-field must go to zero on both sides of the interface at a perfect conductor (σ = ∞). You should identify in your explanation which of Maxwell's laws is responsible. If you wish, you may supplement your explanation with diagrams or equations.

There are multiple ways to think about this problem.



Imagine a time varying B-field line passing through the surface of a perfect conductor. If $\frac{\partial \vec{B}}{\partial t} \neq 0$ then Faraday's Law tells us a solenoidal E-field will drive a current in the conductor. Ampere's Law tells us that this will produce an opposing B-field which cancels B-field inside a perfect conductor.

Time harmonic Faraday's Law: $\nabla \times \tilde{\vec{E}} = -j\omega \tilde{\vec{B}}$

$\tilde{B}_{2n} = 0$ since $\tilde{E} = 0$ inside a perfect conductor – otherwise infinite current would flow. Gauss's Law $\nabla \cdot \tilde{B} = 0$ gives us the boundary condition $\tilde{B}_{1n} = \tilde{B}_{2n}$, which must equal zero.

Slightly different view: Since $\nabla \times \tilde{\vec{E}} = -j\omega \tilde{\vec{B}}$, and $\vec{E}$ is normal to surface, then $\vec{B}$ can have no normal component outside.

Extra: The transverse B-field isn't cancelled, because the surface prevents current from flowing solenoidally to cancel it.

(b) (15 points) Now consider the case where the conductor is not perfect ($\sigma \neq \infty$). For which case is the boundary condition listed in (a) a better approximation: the low frequency case ($\omega \rightarrow 0$), or the high frequency case ($\omega \rightarrow \infty$). Explain why.

For finite σ, finite non-zero E-field is required to drive the solenoidal surface currents that produce the cancelling B-field.

Faraday's Law $\nabla \times \tilde{\vec{E}} = -j\omega \tilde{\vec{B}}$

$$\nabla \times \tilde{\vec{J}} = -j\sigma\omega \tilde{\vec{B}}$$

Therefore, if a certain $\vec{J}$ is needed to cancel out $\vec{B}$, then as σ goes down, ω must go up in order to produce sufficient emf to drive the current.

In otherwords, at high frequencies the screening of B-fields is more effective in a conductor, and B will be closer to zero inside the surface.

We saw this in our example of the metal pipe, where H-field $\rightarrow 0$ inside for $\omega \gg \frac{1}{\tau_m}$, where τ_m is the magnetic diffusion time.

Maxwell's Equations in media:

$$\nabla\cdot\mathbf{D} = \rho_f$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

Auxillary Fields: $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$

In linear media: $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$ $\quad$ $\mathbf{D} = \varepsilon\mathbf{E}$

$\mathbf{M} = \chi_m\mathbf{H}$ $\quad$ $\mathbf{B} = \mu\mathbf{H}$

Ohm's law: $\mathbf{J} = \sigma\mathbf{E}$

Electrostatic Scalar Potential: $\mathbf{E} = -\nabla V$ $\quad$ Vector potential: $\mathbf{B} = \nabla\times\mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla\cdot\mathbf{A})\, dV = \oint_S \mathbf{A}\cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$

Electric energy density: $W_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$ or $W_e = \frac{1}{2}\varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$ or $W_m = \frac{1}{2}\mu H^2$ (in linear media)

Joule power dissipation density: $W_P = \mathbf{E}\cdot\mathbf{J}$ or $W_m = \sigma E^2$ (in Ohm's law media)

Poynting Vector: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\}$

Capacitance: $C = \frac{Q}{V}$

Inductance: $L = \frac{\Lambda}{I} = N\frac{\Phi}{I}$

Boundary conditions: $E_{t,2} - E_{t,1} = 0$ $\quad$ $H_{t,1} - H_{t,2} = J_s$

$D_{n,2} - D_{n,1} = \rho_s$ $\quad$ $B_{n,2} - B_{n,1} = 0$

Bound charge: $\rho_{b,v} = -\nabla\cdot\mathbf{P}$ $\quad$ $\rho_{b,s} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Bound current: $\mathbf{J}_{b,v} = \nabla\times\mathbf{M}$ $\quad$ $\mathbf{J}_{b,s} = \mathbf{M}\times\hat{\mathbf{n}}$

Definition of phasor $\tilde{F}$ for time harmonic function *f(t)*:
$$\begin{cases} f(t) = \mathrm{Re}\{\tilde{F}e^{j\omega t}\} = |F|\cos(\omega t + \phi) \\ \tan^{-1}(\phi) = \mathrm{Im}\{\tilde{F}\}/\mathrm{Re}\{\tilde{F}\} \end{cases}$$

Constants (SI units): ε_0=8.85x10^{-12} F/m (or $C^2\,N^{-1}\,m^{-2}$) $\quad$ μ_0=4π x10^{-7} H/m (or N A^{-2})

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{x}A_x + \hat{y}A_y + \hat{z}A_z$	$\hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$	$\hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$
Magnitude of A, $\lvert A\rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$, for $P(x_1, y_1, z_1)$	$\hat{r}r_1 + \hat{z}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{R}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$ $\hat{x}\times\hat{y} = \hat{z}$ $\hat{y}\times\hat{z} = \hat{x}$ $\hat{z}\times\hat{x} = \hat{y}$	$\hat{r}\cdot\hat{r} = \hat{\phi}\cdot\hat{\phi} = \hat{z}\cdot\hat{z} = 1$ $\hat{r}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{z} = \hat{z}\cdot\hat{r} = 0$ $\hat{r}\times\hat{\phi} = \hat{z}$ $\hat{\phi}\times\hat{z} = \hat{r}$ $\hat{z}\times\hat{r} = \hat{\phi}$	$\hat{R}\cdot\hat{R} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$ $\hat{R}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{R} = 0$ $\hat{R}\times\hat{\theta} = \hat{\phi}$ $\hat{\theta}\times\hat{\phi} = \hat{R}$ $\hat{\phi}\times\hat{R} = \hat{\theta}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$	$\hat{r}\,dr + \hat{\phi}r\,d\phi + \hat{z}\,dz$	$\hat{R}\,dR + \hat{\theta}R\,d\theta + \hat{\phi}R\sin\theta\,d\phi$
Differential surface areas	$ds_x = \hat{x}\,dy\,dz$ $ds_y = \hat{y}\,dx\,dz$ $ds_z = \hat{z}\,dx\,dy$	$ds_r = \hat{r}r\,d\phi\,dz$ $ds_\phi = \hat{\phi}\,dr\,dz$ $ds_z = \hat{z}r\,dr\,d\phi$	$ds_R = \hat{R}R^2\sin\theta\,d\theta\,d\phi$ $ds_\theta = \hat{\theta}R\sin\theta\,dR\,d\phi$ $ds_\phi = \hat{\phi}R\,dR\,d\theta$
Differential volume, dv =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$ $\hat{z} = \hat{z}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{x} = \hat{r}\cos\phi - \hat{\phi}\sin\phi$ $\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi$ $\hat{z} = \hat{z}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$ $\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$ $\hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$ $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{R} = \hat{r}\sin\theta + \hat{z}\cos\theta$ $\hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$ $\hat{\phi} = \hat{\phi}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{r} = \hat{R}\sin\theta + \hat{\theta}\cos\theta$ $\hat{\phi} = \hat{\phi}$ $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(U + V) = \nabla U + \nabla V$$

$$\nabla(UV) = U\nabla V + V\nabla U$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$$

$$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times \nabla V = 0$$

$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A})\, d\mathcal{V} = \oint_S \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses $\mathcal{V}$)

$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)