SOLUTIONS

UCLA Department of Electrical Engineering
EE101A – Engineering Electromagnetics
Fall 2016
Midterm, November 1 2016, (1:45 minutes)

Name ______________________________ Student number ______________________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	45	
Problem 2	Electrostatics	10	
Problem 3	Inductance	45	
Total		100	

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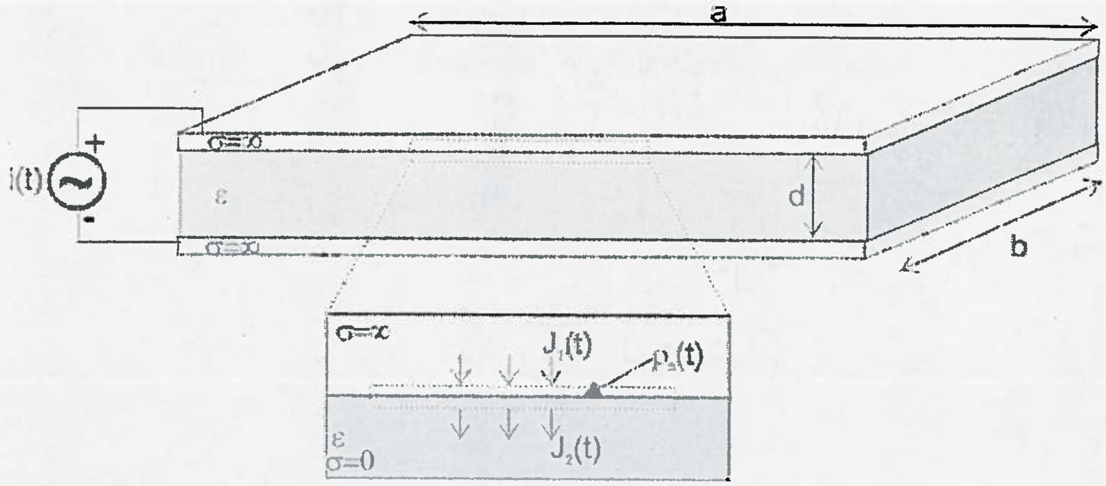
1. Capacitor (45 points)



(a) (15 points) Consider a parallel plate capacitor with plate dimensions a and b, and a perfectly insulating dielectric of thickness d, and a permittivity ε. Assume that the capacitor is hooked up to a current source with time varying current $i(t)=i_0\cos(\omega t)$. The voltage difference between the plates is $v(t)$. The capacitor obeys the standard relation $i(t) = C\, dv(t)/dt$.

Consider a closed Gaussian surface that surrounds the interface between the top metal plate and the dielectric. Write an expression for the current density flowing to the top surface $J_1(t)$, out of the bottom surface $J_2(t)$, and the charge density at the interface $\rho_s(t)$ (all as shown in figure).

The current $i(t)$ that enters the top plate will be spead evenly to create a top current density $J_1(t) = \frac{i(t)}{ab}$

$$\boxed{J_1(t) = \frac{i_0 \cos\omega t}{ab}}$$

Since the dielectric is insulating, $\boxed{J_2(t) = 0}$.

Use current continuity to find $\rho_s(t)$

$$\oint \vec{J}\cdot d\vec{S} = -\frac{dQ_{enc}}{dt}$$

$$\frac{d\rho_s(t)}{dt} = J_1(t) = \frac{i_0 \cos\omega t}{ab}$$

$$\int_{\rho_s(t=0)}^{\rho_s(t)} d\rho_s(t) = \int_0^t \frac{i_0}{ab}\cos\omega t$$

$$\boxed{\rho_s(t) = \rho_s(t=0) + \frac{i_0}{ab\omega}\sin\omega t}$$

we may assume $\rho_s(t=0) = 0$ if we wish.

(b) (15 points) Now we will use phasors to re-solve (a). Assume the current can be written as $i(t) = Re\{\tilde{I}e^{j\omega t}\}$ and $v(t) = Re\{\tilde{V}e^{j\omega t}\}$ and similarly for all other time varying quantities.

Rewrite the capacitor relation $i(t) = C\, dv(t)/dt$, in phasor form for $\tilde{I}, \tilde{V}$.

Give ~~the expressions for the~~ equivalent ~~phasor quantities:~~ $\tilde{J}_1, \tilde{J}_2, \tilde{\rho}_s$ in terms $\tilde{I}$.

$$\tilde{I} = Cj\omega\tilde{V}$$

5 pts

$$\tilde{J}_1 = \frac{\tilde{I}_1}{ab} = \frac{i_0}{ab} \qquad \tilde{J}_2 = 0$$

3 2

$$\tilde{J}_1 = +j\omega\tilde{\rho}_s$$

$$\tilde{\rho}_s = \frac{-j}{\omega}\frac{\tilde{I}_1}{ab}$$

5 pts

Check answer: convert into time domain.

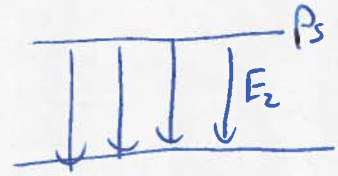
$$\rho_s(t) = Re\{\tilde{\rho}_s e^{j\omega t}\} = Re\left\{\frac{-j\tilde{I}_1}{\omega ab}(\cos\omega t + j\sin\omega t)\right\}$$

$$\rho_s(t) = \frac{i_0}{\omega ab}\sin\omega t \quad \text{it checks out}$$

let $\tilde{I}_1 = i_0$

(c) (15 points) Challenge problem: Now consider that the dielectric is "leaky", i.e. it has a non-zero conductivity σ. Write new expressions for $\tilde{J}_1, \tilde{J}_2, \tilde{\rho}_s$.

Current Continuity tells us: $-\tilde{J}_1 + \tilde{J}_2 = -j\omega\tilde{\rho}_s$



Current in dielectric B $\quad \tilde{J}_2 = \sigma\tilde{E}_2$

E_2 is related to ρ_s : $\quad \tilde{E}_2 = \tilde{\rho}_s/\varepsilon$

so $\quad \tilde{J}_2 = \frac{\sigma}{\varepsilon}\tilde{\rho}_s = \tilde{\rho}_s/\tau_e \qquad \tau_e = \varepsilon/\sigma$ dielectric relax time

Plug into 1st eq:

$$-\tilde{J}_1 + \tilde{\rho}_s/\tau_e = -j\omega\tilde{\rho}_s$$

$$\tilde{\rho}_s = \frac{\tilde{J}_1\tau_e}{1+j\omega\tau_e} \qquad \tilde{J}_1 = \frac{i_0}{ab} \qquad \tilde{J}_2 = \frac{\tilde{J}_1}{1+j\omega\tau_e}$$

2. Electrostatics (10 points)

One of these is an impossible electrostatic field. Which one? You must explain why for credit.

(A): $\mathbf{E} = 4\left[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}\right]$

(B): $\mathbf{E} = 2\left[y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}\right]$

Electrostatic E-fields are conservative and obey

$$\nabla \times \vec{E} = 0$$

$$\nabla \times \vec{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

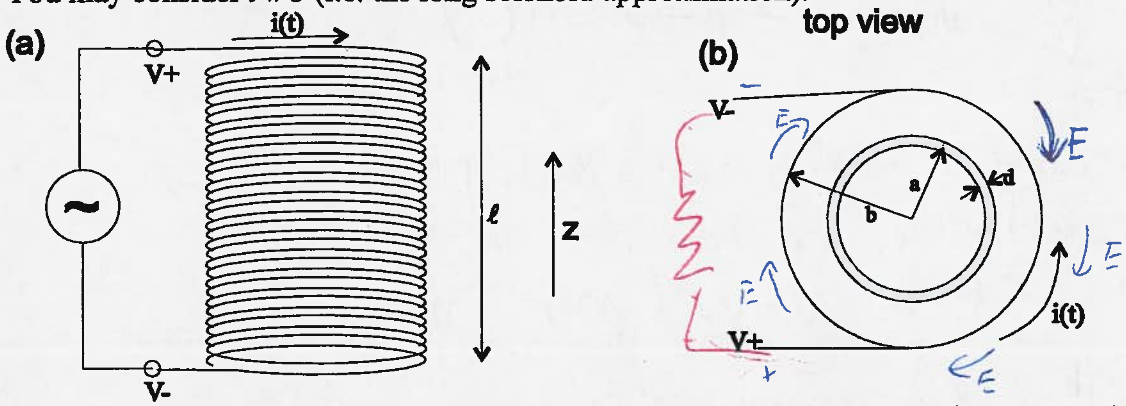
(A) $\nabla \times \vec{E} = 4\left[\hat{x}(0-2y) + \hat{y}(0-3z) + \hat{z}(0-x)\right] \neq 0$

(B) $\nabla \times \vec{E} = 2\left[\hat{x}(2z-2z) + \hat{y}(0-0) + \hat{z}(2y-2y)\right] = 0$

A is not electrostatic

3. Inductance (45 points)

Consider a long solenoid of length l, with N turns, and a radius of b as shown in part (a) of the figure. You may consider $l \gg b$ (i.e. the long solenoid approximation).



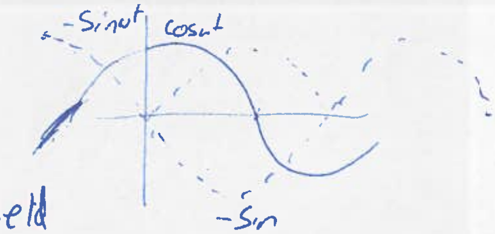
(a) (15 points) What is the self-inductance of a long solenoid (shown in part (a) of the figure), in terms of fundamental constants, and the parameters mentioned above?

$$L = \frac{\mu_0 N^2 \pi b^2}{l}$$

(b) (15 points) Assume a current $i(t)=I_0 \cos(\omega t)$ is flowing as shown. What is the voltage $v(t)=V_+ - V_-$ at the terminals as a function of time (as shown in part (a)? Pay attention to the sign.

$V = L \frac{di}{dt}$ $V(t) = -I_0 L \omega \sin(\omega t)$

$= -V_{emf}$



When current is increasing, a solenoidal E-field is created in the $-\hat{\phi}$ direction. This causes a positive Voltage $V_+ - V_-$ to appear. This is consistent with $V(t) = -I_0 L \omega \sin \omega t$.

Now imagine a piece of superconducting pipe that is a perfect electrical conductor with radius $a=b/2$, and thickness d is inserted into the center of the solenoid, as shown in part (b) of the figure.

(c) (15 points) Will the addition of the perfectly conducting ($\sigma = \infty$) pipe increase, decrease, or leave unchanged the apparent self-inductance L of the solenoid? Give a qualitative explanation why.

The self inductance will decrease due to the reduction of flux inside the pipe. Since the pipe is a perfect conductor, it will perfectly screen out applied B-fields, so that $B=0$ inside $(r < a)$.

$L = \frac{\Phi}{I}$

Maxwell's Equations in media:

$$\nabla\cdot\mathbf{D} = \rho_f$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

Auxillary Fields:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In linear media: $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$ $\quad \mathbf{D} = \varepsilon\mathbf{E}$

$\mathbf{M} = \chi_m\mathbf{H}$ $\quad \mathbf{B} = \mu\mathbf{H}$

Ohm's law: $\mathbf{J}_f = \sigma\mathbf{E}$

Electrostatic Scalar Potential: $\mathbf{E} = -\nabla V$ Vector potential: $\mathbf{B} = \nabla\times\mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla\cdot\mathbf{A})\,dV = \oint_S \mathbf{A}\cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$

Electric energy density: $W_e = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$ or $W_e = \frac{1}{2}\varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$ or $W_m = \frac{1}{2}\mu H^2$ (in linear media)

Joule power dissipation density: $W_P = \mathbf{E}\cdot\mathbf{J}$ or $W_m = \sigma E^2$ (in Ohm's law media)

Poynting Vector: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\}$

Capacitance: $C = \frac{Q}{V}$

Inductance: $L = \frac{\Lambda}{I} = N\frac{\Phi}{I}$

Boundary conditions: $E_{t,2} - E_{t,1} = 0$ $\quad H_{t,1} - H_{t,2} = J_s$

$D_{n,2} - D_{n,1} = \rho_s$ $\quad B_{n,2} - B_{n,1} = 0$

Bound charge: $\rho_{b,v} = -\nabla\cdot\mathbf{P}$ $\quad \rho_{b,s} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Bound current: $\mathbf{J}_{b,v} = \nabla\times\mathbf{M}$ $\quad \mathbf{J}_{b,s} = \mathbf{M}\times\hat{\mathbf{n}}$

Definition of phasor $\tilde{F}$ for time harmonic function $f(t)$:

$$\begin{cases} f(t) = \mathrm{Re}\{\tilde{F}e^{j\omega t}\} = |F|\cos(\omega t + \phi) \\ \tan^{-1}(\phi) = \mathrm{Im}\{\tilde{F}\}/\mathrm{Re}\{\tilde{F}\} \end{cases}$$

Constants (SI units): ε_0=8.85x10^{-12} F/m (or C^2 N^{-1} m^{-2}) $\quad \mu_0$=4π x10^{-7} H/m (or N A^{-2})

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x,y,z	r,ϕ,z	R,θ,ϕ
Vector representation, A =	$\hat{x}A_x+\hat{y}A_y+\hat{z}A_z$	$\hat{r}A_r+\hat{\phi}A_\phi+\hat{z}A_z$	$\hat{R}A_R+\hat{\theta}A_\theta+\hat{\phi}A_\phi$
Magnitude of A, $\|A\|$ =	$\sqrt[+]{A_x^2+A_y^2+A_z^2}$	$\sqrt[+]{A_r^2+A_\phi^2+A_z^2}$	$\sqrt[+]{A_R^2+A_\theta^2+A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{x}x_1+\hat{y}y_1+\hat{z}z_1$, for $P(x_1,y_1,z_1)$	$\hat{r}r_1+\hat{z}z_1$, for $P(r_1,\phi_1,z_1)$	$\hat{R}R_1$, for $P(R_1,\theta_1,\phi_1)$
Base vectors properties	$\hat{x}\cdot\hat{x}=\hat{y}\cdot\hat{y}=\hat{z}\cdot\hat{z}=1$ $\hat{x}\cdot\hat{y}=\hat{y}\cdot\hat{z}=\hat{z}\cdot\hat{x}=0$ $\hat{x}\times\hat{y}=\hat{z}$ $\hat{y}\times\hat{z}=\hat{x}$ $\hat{z}\times\hat{x}=\hat{y}$	$\hat{r}\cdot\hat{r}=\hat{\phi}\cdot\hat{\phi}=\hat{z}\cdot\hat{z}=1$ $\hat{r}\cdot\hat{\phi}=\hat{\phi}\cdot\hat{z}=\hat{z}\cdot\hat{r}=0$ $\hat{r}\times\hat{\phi}=\hat{z}$ $\hat{\phi}\times\hat{z}=\hat{r}$ $\hat{z}\times\hat{r}=\hat{\phi}$	$\hat{R}\cdot\hat{R}=\hat{\theta}\cdot\hat{\theta}=\hat{\phi}\cdot\hat{\phi}=1$ $\hat{R}\cdot\hat{\theta}=\hat{\theta}\cdot\hat{\phi}=\hat{\phi}\cdot\hat{R}=0$ $\hat{R}\times\hat{\theta}=\hat{\phi}$ $\hat{\theta}\times\hat{\phi}=\hat{R}$ $\hat{\phi}\times\hat{R}=\hat{\theta}$
Dot product, A · B =	$A_xB_x+A_yB_y+A_zB_z$	$A_rB_r+A_\phi B_\phi+A_zB_z$	$A_RB_R+A_\theta B_\theta+A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{r} & \hat{\phi} & \hat{z}\\ A_r & A_\phi & A_z\\ B_r & B_\phi & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{R} & \hat{\theta} & \hat{\phi}\\ A_R & A_\theta & A_\phi\\ B_R & B_\theta & B_\phi\end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{x}dx+\hat{y}dy+\hat{z}dz$	$\hat{r}dr+\hat{\phi}rd\phi+\hat{z}dz$	$\hat{R}dR+\hat{\theta}Rd\theta+\hat{\phi}R\sin\theta\, d\phi$
Differential surface areas	$ds_x=\hat{x}dydz$ $ds_y=\hat{y}dxdz$ $ds_z=\hat{z}dxdy$	$ds_r=\hat{r}rd\phi dz$ $ds_\phi=\hat{\phi}drdz$ $ds_z=\hat{z}rdrd\phi$	$ds_R=\hat{R}R^2\sin\theta d\theta d\phi$ $ds_\theta=\hat{\theta}R\sin\theta dRd\phi$ $ds_\phi=\hat{\phi}RdRd\theta$
Differential volume, $d\mathcal{V}$ =	$dxdydz$	$rdrd\phi dz$	$R^2\sin\theta dRd\theta d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r=\sqrt[+]{x^2+y^2}$ $\phi=\tan^{-1}(y/x)$ $z=z$	$\hat{r}=\hat{x}\cos\phi+\hat{y}\sin\phi$ $\hat{\phi}=-\hat{x}\sin\phi+\hat{y}\cos\phi$ $\hat{z}=\hat{z}$	$A_r=A_x\cos\phi+A_y\sin\phi$ $A_\phi=-A_x\sin\phi+A_y\cos\phi$ $A_z=A_z$
Cylindrical to Cartesian	$x=r\cos\phi$ $y=r\sin\phi$ $z=z$	$\hat{x}=\hat{r}\cos\phi-\hat{\phi}\sin\phi$ $\hat{y}=\hat{r}\sin\phi+\hat{\phi}\cos\phi$ $\hat{z}=\hat{z}$	$A_x=A_r\cos\phi-A_\phi\sin\phi$ $A_y=A_r\sin\phi+A_\phi\cos\phi$ $A_z=A_z$
Cartesian to spherical	$R=\sqrt[+]{x^2+y^2+z^2}$ $\theta=\tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi=\tan^{-1}(y/x)$	$\hat{R}=\hat{x}\sin\theta\cos\phi+\hat{y}\sin\theta\sin\phi+\hat{z}\cos\theta$ $\hat{\theta}=\hat{x}\cos\theta\cos\phi+\hat{y}\cos\theta\sin\phi-\hat{z}\sin\theta$ $\hat{\phi}=-\hat{x}\sin\phi+\hat{y}\cos\phi$	$A_R=A_x\sin\theta\cos\phi+A_y\sin\theta\sin\phi+A_z\cos\theta$ $A_\theta=A_x\cos\theta\cos\phi+A_y\cos\theta\sin\phi-A_z\sin\theta$ $A_\phi=-A_x\sin\phi+A_y\cos\phi$
Spherical to Cartesian	$x=R\sin\theta\cos\phi$ $y=R\sin\theta\sin\phi$ $z=R\cos\theta$	$\hat{x}=\hat{R}\sin\theta\cos\phi+\hat{\theta}\cos\theta\cos\phi-\hat{\phi}\sin\phi$ $\hat{y}=\hat{R}\sin\theta\sin\phi+\hat{\theta}\cos\theta\sin\phi+\hat{\phi}\cos\phi$ $\hat{z}=\hat{R}\cos\theta-\hat{\theta}\sin\theta$	$A_x=A_R\sin\theta\cos\phi+A_\theta\cos\theta\cos\phi-A_\phi\sin\phi$ $A_y=A_R\sin\theta\sin\phi+A_\theta\cos\theta\sin\phi+A_\phi\cos\phi$ $A_z=A_R\cos\theta-A_\theta\sin\theta$
Cylindrical to spherical	$R=\sqrt[+]{r^2+z^2}$ $\theta=\tan^{-1}(r/z)$ $\phi=\phi$	$\hat{R}=\hat{r}\sin\theta+\hat{z}\cos\theta$ $\hat{\theta}=\hat{r}\cos\theta-\hat{z}\sin\theta$ $\hat{\phi}=\hat{\phi}$	$A_R=A_r\sin\theta+A_z\cos\theta$ $A_\theta=A_r\cos\theta-A_z\sin\theta$ $A_\phi=A_\phi$
Spherical to cylindrical	$r=R\sin\theta$ $\phi=\phi$ $z=R\cos\theta$	$\hat{r}=\hat{R}\sin\theta+\hat{\theta}\cos\theta$ $\hat{\phi}=\hat{\phi}$ $\hat{z}=\hat{R}\cos\theta-\hat{\theta}\sin\theta$	$A_r=A_R\sin\theta+A_\theta\cos\theta$ $A_\phi=A_\phi$ $A_z=A_R\cos\theta-A_\theta\sin\theta$

GRADIENT, DIVERGENCE, CURL & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB\sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing $\mathbf{A}$ and $\mathbf{B}$

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$\nabla(U + V) = \nabla U + \nabla V$

$\nabla(UV) = U\nabla V + V\nabla U$

$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$

$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$

$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$

$\nabla \times \nabla V = 0$

$\nabla \cdot \nabla V = \nabla^2 V$

$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A})\, d\mathcal{V} = \oint_S \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses $\mathcal{V}$)

$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)

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