UCLA Department of Electrical Engineering
EE101 – Engineering Electromagnetics
Winter 2013
Final Exam, March 18 2013, (3 hours)

Name ______________________________ Student number______________________

This is a closed book exam – you are allowed 2 page of notes (each page front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit.

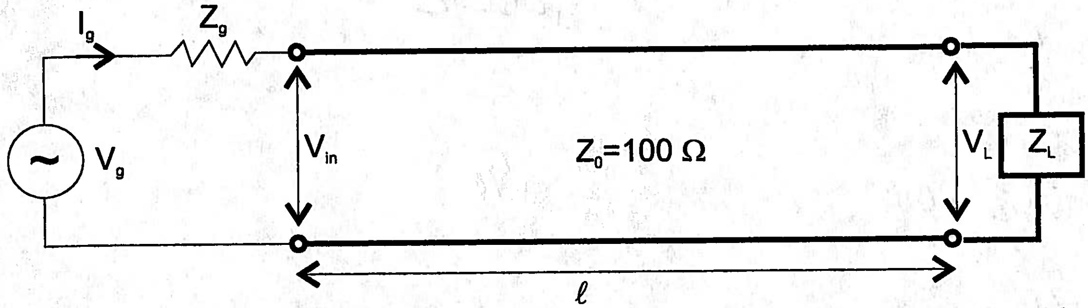
If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Smith Chart	15	
Problem 2	Impedance Matching	30	
Problem 3	Plane wave	30	
Problem 4	Phasors	25	
Total		100	

1. Smith chart basics (15 points)

Consider the generic transmission line problem as shown below. Assume that the transmission line is a coaxial transmission line filled with a material that has $\varepsilon=4\varepsilon_0$, $\mu=\mu_0$.



(a) (5 points) For each of the following loads, mark their position on the Smith chart below (using the letter as a label), and write below the reflection coefficient (magnitude and phase angle).

A: Z_L= 150 Ω. Γ = 0.2

$z_L = 1.5$

B: Z_L = 200+*j80* Ω. Γ = $0.41 \angle 24°$

$z_L = 2+0.8j$

(b) (5 points) For each of the following loads impedances, convert to unnormalized load admittance Y_L and give the value in units Ω^{-1}. Mark the position on the Smith chart below (using the letter as a label).

$Y_0 = 0.01$

A: Z_L= 150 Ω. A' Y_L= $0.0065\ \Omega^{-1}$

$g_L = 0.65$

B: Z_L = 200+*j80* Ω. B' Y_L=

$y_L = 0.43 - 0.17j$

$0.0043 - j0.0017\ \Omega^{-1}$

(c) (5 points) What is the non-normalized input impedance of the transmission line $Z_{in}(-l)$ for each of the loads if l=1 m and f=30 MHz? Label each point on the Smith Chart using A'', B''.

$\ell = 1.0\,m$

A: Z_L= 150 Ω. A'' Z_{in} = $70 - j20\,\Omega$

$z_L = 0.7 - j0.2$

B: $Z_L = 200+j80\ \Omega$. B'' Z_{in} = $52 - j46\,\Omega$

$z_L = 0.52 - j0.46$

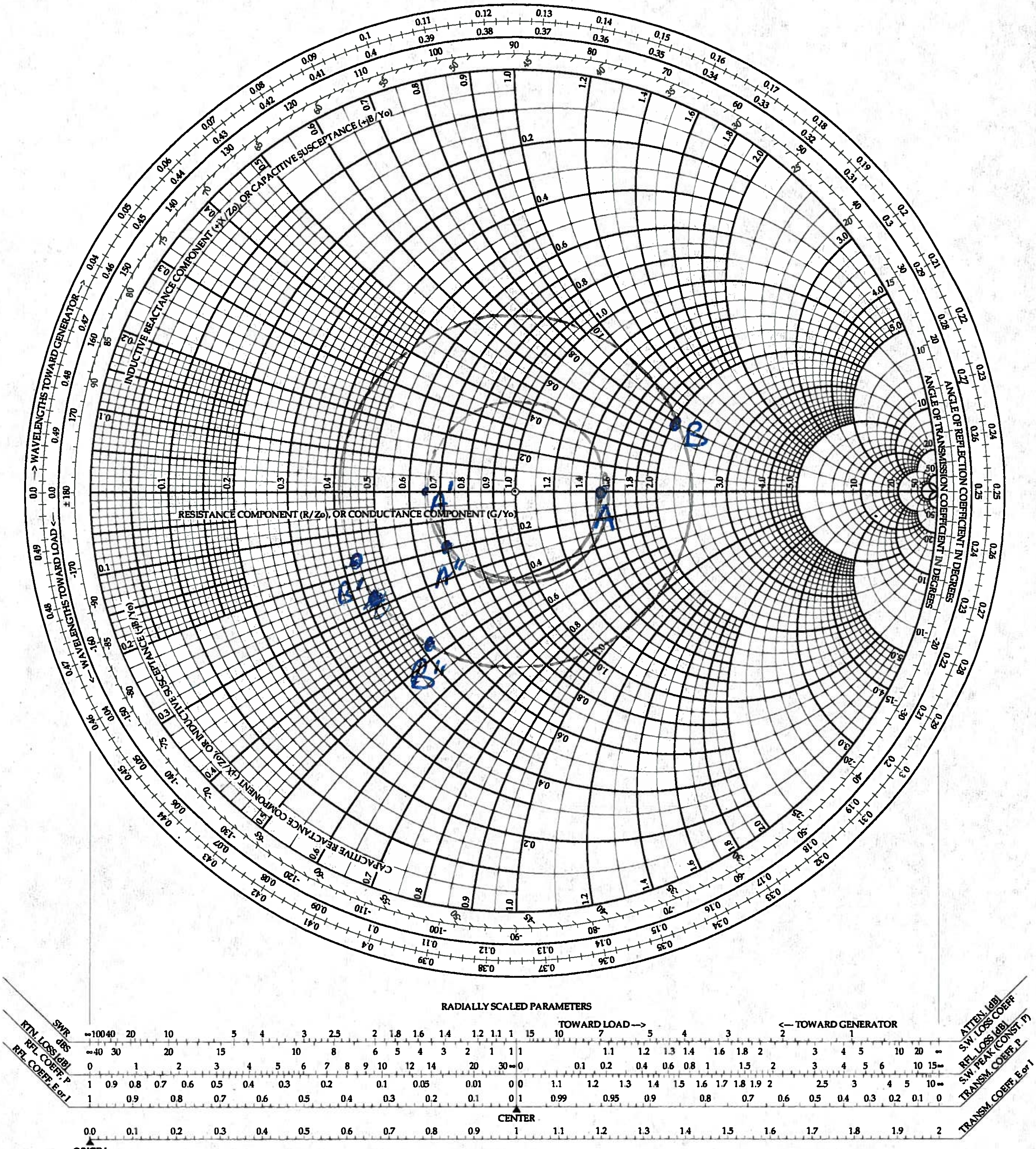
if $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$

$$V = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{2} = 1.5\times10^8\ m/s$$

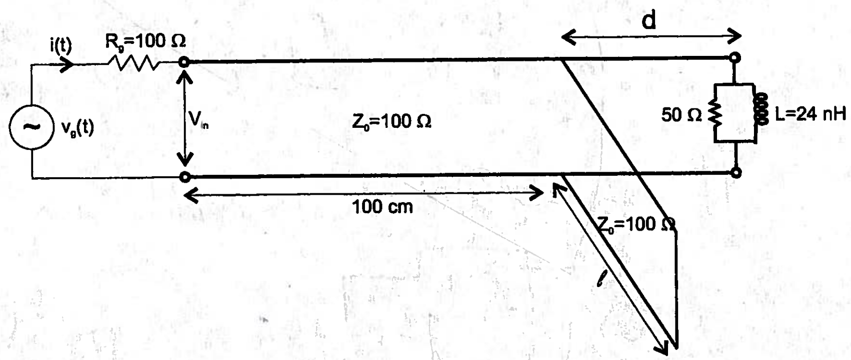
$$\lambda = \frac{1.5\times10^8\ m/s}{3\times10^7\ s^{-1}} = 0.5\times10 = 5\ m$$

$$\ell = \frac{1.0}{5}\lambda = 0.2\lambda$$

Smith chart for problem 1



2. **Transmission line – Impedance Matching** **(30 points)**
This problem involves a transmission line which is lossless coaxial cable filled with a dielectric material $\varepsilon=9\varepsilon_0$, $\mu=\mu_0$. The frequency of operation is f=200 MHz.
For these problems, you may use any methods you wish, including the Smith chart (not required).



(a) (10 points) At the frequency given above what is the load impedance in units Ω (give a number) ?

$f = 200\,MHz = 2\times10^8\,Hz$ $\quad W = 1.26\times10^9\,s^{-1}$ $\quad Z_{ind} = jWL = j30\,\Omega$

$Y_L = \frac{1}{50} - j\frac{1}{30} = 0.02 - j0.033\,\Omega^{-1}$ $\quad Y_L =$

$Z_L = 13 + j22\,\Omega$

(b) (10 points) Use a shorted stub to match the load to the feedline and prevent any reflections back down the feedline. What values of d and l should be chosen to achieve matching to the line? Give these length values in term of the wavelength λ.

Two answers

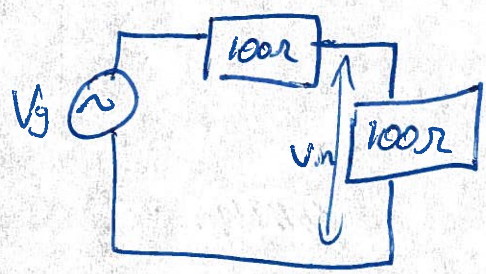
① $d = 0.02\lambda$, $l = 0.437\lambda$

② $d = 0.41\lambda$ $\quad l = 0.064\lambda$

(c) (10 points) If the generator voltage is given by $v_g(t)=v_0 \cos \omega t$, where $v_0=1$ V, what is the time averaged power dissipated in the load? Give a number in units W.

$$P_{avg} = \frac{1}{2} Re\{V_{in} I^*_{in}\} = \frac{1}{2} |V_{in}|^2 Re\left\{\frac{1}{Z_{in}}\right\}$$

If $Z_{in} = 100\,\Omega$ in the matched case



$$V_{in} = \frac{V_g}{2}$$

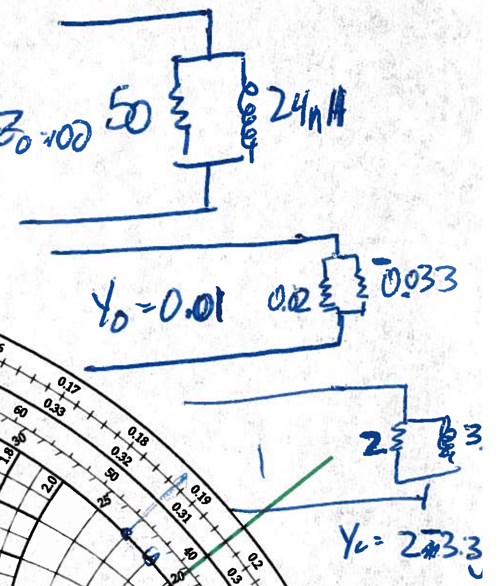
$$\boxed{P_{avg} = \frac{1}{2} \frac{(1V)^2}{4} \frac{1}{100} = \frac{1}{800}\,W}$$

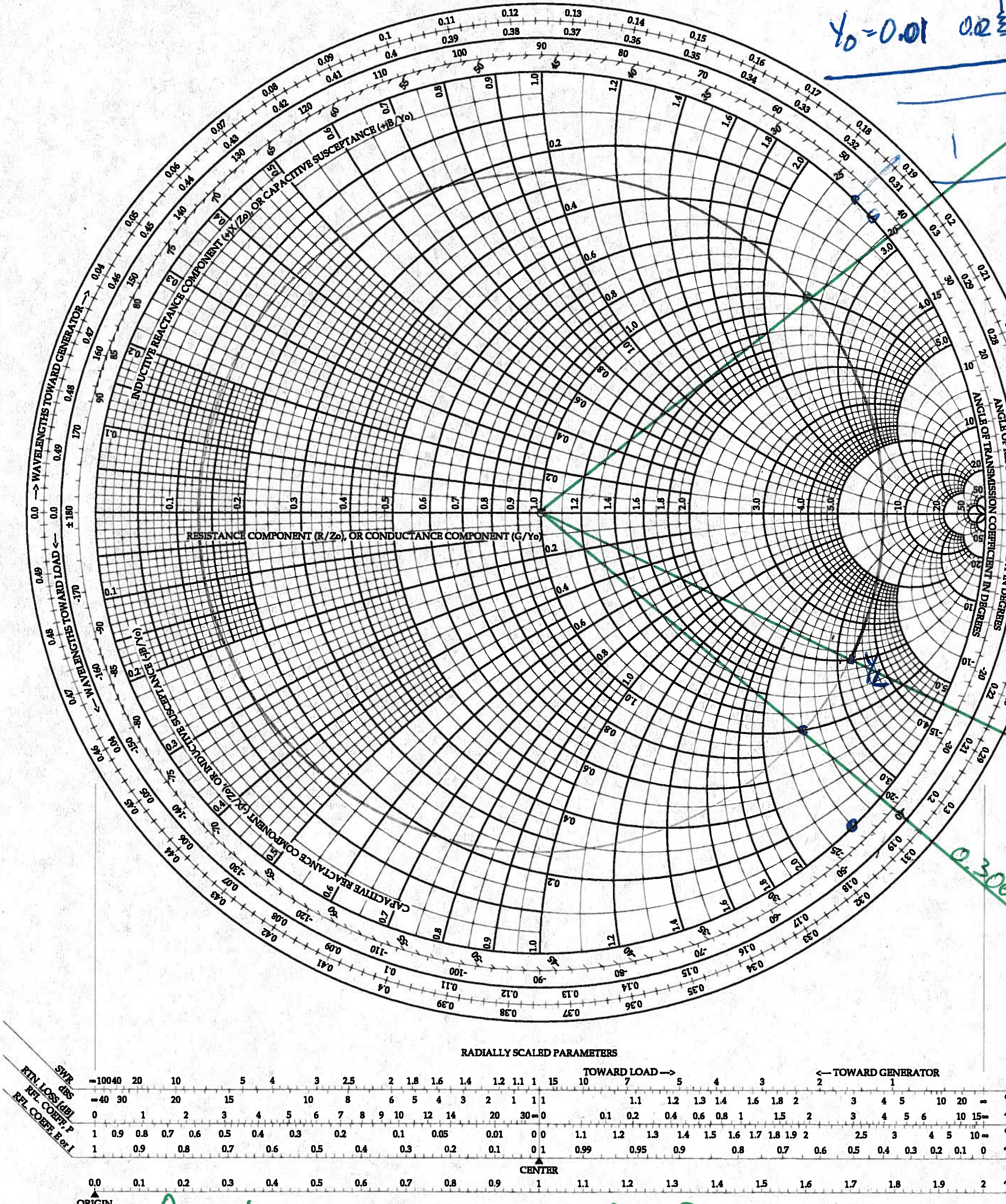
$$= 0.00125\,W$$

The Complete Smith Chart

Black Magic Design

$Y_{ind} = \frac{1}{j\omega L} =$





Answer 1

$d = 0.02\lambda$

$l = \cancel{0.\ldots}\ 0.437\ldots$

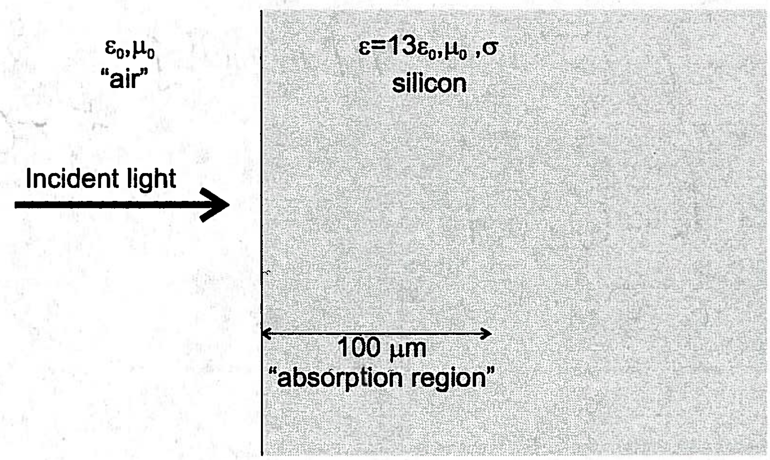
Answer 2

d= ~~[illegible]~~ $d = 0.41\lambda$

$l = 0.064\lambda$

3. **Plane waves and interfaces** **(30 points)**

Consider that we are building a solar cell out of silicon ($\varepsilon = 13\varepsilon_0$), where the absorbing layer is 100 μm thick. Let's approximate this by a model where a plane wave with a free-space wavelength of λ=1 μm is normally incident on the surface of the silicon (which we take to be semi-infinite, with permittivity $\epsilon=\epsilon_0$, $\mu=\mu_0$). Our goal is to have the maximum fraction of incident light be transmitted, and then be absorbed within the first 100 μm (whereupon it is presumably converted into electrical power). Assume that due to the design of the solar cell junction, any light that passes beyond 100μm from the surface is not collected and thus is wasted.



We have the choice of 3 possible values of conductivity σ for the silicon.
Case A: $\sigma=10^7$ S/m, Case B: $\sigma=10^3$ S/m, Case C: $\sigma=10^{-1}$ S/m

(a) (6 points) For each case, state whether the silicon can be considered a good conductor, or a poor conductor (aka lossy dielectric).

$\varepsilon' = 13\varepsilon_0$ $\varepsilon'' = \frac{\sigma}{\omega}$ $\lambda = 1\mu m$, $f = 300 THz$, $\omega = 1.9\times10^{15} s^{-1}$

$\varepsilon' = 1.2\times10^{-10} F/m$

Case A:

$\varepsilon'' \approx 5\times10^{-9}$ $\frac{\varepsilon''}{\varepsilon'} \approx 45 \gg 1$ so it is a "good conductor".

Case B:

$\varepsilon'' \approx 5\times10^{-3}$ $\frac{\varepsilon''}{\varepsilon'} \approx 0.004 \ll 1$ $\Rightarrow$ lossy dielectric.

Case C:

$\varepsilon'' \approx 5\times10^{-17}$ $\frac{\varepsilon''}{\varepsilon'} \ll 1$ lossy dielectric

(b) (9 points) For each case, calculate the reflection coefficient Γ, and the power reflectivity R.

Good conductor case:

$$\eta_c = \sqrt{\frac{\mu_0}{\varepsilon_c}} \approx \sqrt{\frac{\mu_0 \omega}{-j\omega\sigma}} = \frac{(1+j)}{\sqrt{2}}\sqrt{\frac{\mu_0\omega}{\sigma}} \quad \text{Case A: } (1+j)11\ \Omega$$

Dielectric case

$$\eta \approx \sqrt{\frac{\mu_0}{\varepsilon'}} = \frac{377}{\sqrt{13}} = 105\,\Omega$$

Case A: $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -0.94 + 0.05j = 0.94 \angle 177^\circ \approx -0.94$

R=

$R = 0.89$

Case B: $\Gamma = -0.56$

lossy dielectric

$R = 0.32$

Case C: $\Gamma = -0.56$

$R = 0.32$

(c) (10 points) Which case for the conductivity should we choose to maximize the fraction of incident power that is absorbed by the silicon within the first 100 μm? Why (explanation required for credit) ?

$\boxed{\text{Case B}}$

Case A is too reflective, due to the fact that the silicon is a good conductor. 89% of the incident light will be reflected, which is not efficient.

Case B will have 68% of the light transmitted through the interface. We can calculate the attenuation length $\frac{1}{\alpha}$:

$$\gamma = \alpha + j\beta = \sqrt{-\omega^2 \varepsilon \mu_0} = \sqrt{-\omega^2 \mu_0 (\varepsilon' - j\varepsilon'')}$$

$$= j\omega\sqrt{\mu_0 \varepsilon' \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)}$$

Since $\varepsilon' >> \varepsilon''$.

$\varepsilon'' = \sigma/\omega$

$$\approx j\omega\sqrt{\mu_0\varepsilon'}\left(1 - \frac{j}{2}\frac{\varepsilon''}{\varepsilon'}\right)$$

$$= \frac{\omega\sqrt{\mu_0\varepsilon'}}{2}\frac{\varepsilon''}{\varepsilon'} + j\omega\sqrt{\mu_0}$$

$$\alpha = \frac{\omega\sqrt{\mu_0\varepsilon'}}{2}\frac{\sigma}{\varepsilon'\omega} = \sqrt{\frac{\mu_0}{\varepsilon'}}\frac{\sigma}{2} = 5.25\times10^{4}\,m^{-1}$$

$$\frac{1}{\alpha} = 1.9\times10^{-5}\,m = 19\,\mu m$$

Since the absorbing layer is 100 um thick, we can confidently say that >99% of the power will be absorbed within that layer.

Case C $\alpha = 5.25\,m^{-1}$ $\frac{1}{\alpha} = 0.19\,m. >> 100\,\mu m$

Only a small fraction of the power will be absorbed within the absorbing layer.

$\boxed{\text{So case B is the best choice}}$

(d) (5 points) The performance of this solar cell can be easily improved if we add an anti-reflective thin layer to the surface of the silicon. How thick should this layer be (in meters), and what value of permittivity (or refractive index) should it have?

Use a quarter wave layer.

Choose $\boxed{n = \sqrt{\sqrt{13}} = 1.9}$

Within that material $\lambda = \dfrac{\lambda_0}{n} = \dfrac{1\,um}{1.9} = 0.526\,um$

$\boxed{\text{So we choose thickness} = \dfrac{\lambda}{4} = 0.132\,um = 132\,nm}$

4. Phasors **(25 points)**

(a) (5 points) Write the following phasor quantity explicitly in the time domain (i.e. do not include the expression "Re{ }" in your answer). Assume E_0 is a real number.

$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_0 e^{-jkz}$ $\quad$ $\mathbf{E}(z,t) = \hat{x} E_0 \cos(\omega t - kz)$

(b) (5 points) For the following time dependent signal, find the corresponding phasor expression. (H_0 is real)

$\tilde{\mathbf{H}}(z,t) = \hat{\mathbf{y}} H_0 \sin(\omega t - kz)$ $\quad$ $\tilde{\mathbf{H}}(z) = -\hat{y} j H_0 e^{-jkz}$

$\cos(\omega t - kz - \frac{\pi}{2})$

(c) (5 points) Consider a circularly polarized wave in vacuum with E-field given by

$\mathbf{E}(z,t) = (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) E_0 \cos(\omega t - kz)$

If E_0=1 V/m, f=1 GHz, and the wave is in vacuum, what is the time averaged power flow per unit area for this wave? (Give a number in units W/m^2)

$\frac{1}{2} Re\{\tilde{E} \times \tilde{H}^*\} = \frac{1}{2} Re\{(\hat{x} + j\hat{y}) \cdot E_0 \cdot e^{-jkz} \times [(\hat{y} - j\hat{x}) \cdot \frac{1}{\eta} E_0 e^{-jkz}]^*\}$

$= \frac{1}{\eta} |E_0|^2 \cdot \frac{1}{2} Re\{\hat{z} + \hat{z}\}$

$= \hat{z} \cdot \frac{1}{\eta} |E_0|^2 = \hat{z} \cdot \frac{1}{377} = 2.7 \text{ mW/m}^2$

(d) (5 points) Consider a linearly polarized plane wave in good conductor where $\epsilon \approx -j\sigma/\omega$. The electric field is given by the expression $\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0\cos(\omega t - \beta z)e^{-z/\delta_s}$. Write the expression for the time dependent magnetic field $\mathbf{H}(z,t)$ in terms of E_0, ω, β, δ_s, σ.

$$\vec{H} = \hat{y}\cdot\frac{1}{\eta_c}E_0\cos(\omega t-\beta z)e^{-z/\delta_s} \qquad \eta_c = \sqrt{\frac{\mu_0}{-j\frac{\sigma}{\omega}}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu_0}{\sigma}} = (1+j)\cdot\frac{\cancel{\alpha}}{\sigma}$$

$$= \hat{y}\cdot\frac{1-j}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\mu_0}}E_0\cos(\omega t-\beta z)e^{-z/\delta_s} \qquad \frac{1}{\eta_c} = \sqrt{\frac{-j\frac{\sigma}{\omega}}{\mu_0}} = \frac{1-j}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\mu_0}}$$

$$= \hat{y}\cdot\frac{1-j}{\sqrt{2}}\cdot\frac{\sigma}{\sqrt{2}}\delta_s E_0\cos(\omega t-\beta z)e^{-z/\delta_s}$$

$$\gamma = j\omega\sqrt{\mu\epsilon} = j\omega\frac{1-j}{\sqrt{2}}\sqrt{\mu\frac{\sigma}{\omega}}$$

$$= (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

(e) (5 points) If E_0=1 V/m, f=1 GHz, σ =1S/m, $\mu=\mu_0$, what is the time averaged power flow per unit area for this wave? (Give a number, in units W/m^2).

At z=0

$$\frac{1}{2}Re\{\tilde{E}\times\tilde{H}^*\}\Big|_{z=0} = \frac{1}{2}Re\left\{\hat{x}E_0e^{-j\beta z}e^{-z/\delta_s}\times\left(\hat{y}\cdot\frac{1-j}{\sqrt{2}}\sqrt{\frac{\sigma}{\omega\mu_0}}E_0e^{-j\beta z}e^{-z/\delta_s}\right)^*\right\}$$

+1

$$= \left(\frac{1}{2}\hat{z}\cdot|E_0|^2e^{-2z/\delta_s}\sqrt{\frac{\sigma}{\omega\mu_0}}\cdot\frac{1}{\sqrt{2}}\right)\Big|_{z=0}$$

$$= \hat{z}\cdot\frac{1}{2}\cdot 1\cdot 1\cdot\sqrt{\frac{1}{2\pi\times10^9\times4\pi\times10^{-7}}}\times\frac{1}{\sqrt{2}}$$

$$= \hat{z}\cdot\frac{1}{2}\cdot\frac{1}{2\pi\times10}\times\frac{1}{2}$$

$$= \hat{z}\cdot\frac{1}{80\pi} \cong 3.98\ mW/m^2$$

Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Auxillary Fields:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In linear media: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ $\quad \mathbf{D} = \varepsilon \mathbf{E}$ $\quad \mathbf{M} = \chi_m \mathbf{H}$ $\quad \mathbf{B} = \mu \mathbf{H}$

Ohm's Law: $\mathbf{J} = \sigma \mathbf{E}$ $\quad \mathbf{D} = \varepsilon \mathbf{E}$ $\quad \mathbf{B} = \mu \mathbf{H}$

Electrostatic Potential: $\mathbf{E} = -\nabla V$ $\qquad$ Vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla \cdot \mathbf{A})\, dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$

Electric energy density: $W_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$ or $W_e = \frac{1}{2} \varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \frac{1}{2} \mathbf{B} \cdot \mathbf{H}$ or $W_m = \frac{1}{2} \mu H^2$ (in linear media)

Power dissipation density (Joule/Ohmic) = $\mathbf{E} \cdot \mathbf{J}$ or σE^2 (in Ohm's law media)

Poynting Theorem: $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) - \mathbf{E} \cdot \mathbf{J}$

Poynting Vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \frac{1}{2} \mathrm{Re}\{\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\}$

Capacitance: $C = \frac{Q}{V}$

Inductance: $L = \frac{\Lambda}{I} = N \frac{\Phi}{I}$

Vector identities

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$
$$\nabla \times (\nabla f) = 0$$
$$\nabla \cdot \nabla f = \nabla^2 f$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$
$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

EM waves - Wave equation in source free medium, in time-domain and in harmonic (phasor) form

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{H} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \qquad \nabla^2 \tilde{\mathbf{E}} - \gamma^2 \tilde{\mathbf{E}} = 0 \qquad \nabla^2 \tilde{\mathbf{H}} - \gamma^2 \tilde{\mathbf{H}} = 0$$

$$\gamma^2 = -\omega^2 \mu\varepsilon \qquad \gamma = \alpha + j\beta \qquad k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} = \frac{n\omega}{c} \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

z-propagating Plane wave (linearly polarized in x-direction) – phasor format – nonconducting media

$$\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0}^{+} e^{-jkz} \qquad \tilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_{x0}^{+}}{\eta} e^{-jkz}$$

Generalized plane wave in arbitrary direction with wavevector **k** with arbitrary linear polarization **e**.

$$\tilde{\mathbf{E}}(\mathbf{R}) = \hat{\mathbf{e}} E_0^{+} e^{-j\mathbf{k}\cdot\mathbf{R}} \qquad \tilde{\mathbf{E}} = -\eta \mathbf{k} \times \tilde{\mathbf{H}} \qquad \tilde{\mathbf{H}} = \frac{1}{\eta} \mathbf{k} \times \tilde{\mathbf{E}}$$

Conducting media

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \qquad \gamma = \alpha + j\beta \qquad \eta = \sqrt{\frac{\mu}{\varepsilon_c}} \qquad \delta_s = \frac{1}{\sqrt{\pi f \mu\sigma}} \qquad \tilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0}^{+} e^{-\gamma z} \qquad \tilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_{x0}^{+}}{\eta} e^{-\gamma z}$$

$$\alpha = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right\}^{1/2} \qquad \beta = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right\}^{1/2}$$

Transmission lines

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z) \qquad \frac{d^2\tilde{V}(z)}{dz^2} - \gamma\tilde{V}(z) = 0 \qquad \gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z) \qquad \frac{d^2\tilde{I}(z)}{dz^2} - \gamma\tilde{I}(z) = 0 \qquad Z_0 = \sqrt{\frac{(R' + j\omega L')}{(G' + j\omega C')}}$$

Lossless transmission lines:

$$\gamma = j\beta = j\omega\sqrt{L'C'} \qquad Z_0 = \sqrt{\frac{L'}{C'}} \qquad u_p = \frac{1}{\sqrt{L'C'}} \qquad \beta = \frac{2\pi}{\lambda} = \frac{\omega}{u_p}$$

TEM lossless transmission lines:

$$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon} \qquad Z_0 = \sqrt{\frac{L'}{C'}} \qquad u_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

Transmission line wave solutions (lossless lines)

$$\tilde{V}(z) = V_0^{+} e^{-j\beta z} + V_0^{-} e^{j\beta z} \qquad \tilde{I}(z) = \frac{V_0^{+}}{Z_0} e^{-j\beta z} - \frac{V_0^{-}}{Z_0} e^{j\beta z}$$

$$\Gamma = \frac{V_0^{-}}{V_0^{+}} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad VSWR = \frac{|V_{\max}|}{|V_{\min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{1 + \Gamma e^{j2\beta z}}{1 - \Gamma e^{j2\beta z}} \qquad Z_{in}(z = -l) = Z_0 \frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

Impedance: $Z = R + jX = \dfrac{1}{Y} = \dfrac{1}{G + jB}$ Admittance: $Y = G + jB$

Constants (SI units): $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m (or $C^2\ N^{-1}\ m^{-2}$) $\mu_0 = 4\pi \times 10^{-7}$ H/m (or $N\ A^{-2}$)

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A\rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, dv =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing **A** and **B**

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$\nabla(U + V) = \nabla U + \nabla V$

$\nabla(UV) = U\nabla V + V\nabla U$

$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$

$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$

$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$

$\nabla \times \nabla V = 0$

$\nabla \cdot \nabla V = \nabla^2 V$

$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$\int_{v} (\nabla \cdot \mathbf{A})\, dv = \oint_{S} \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses v)

$\int_{S} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{C} \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)