EE101A – Engineering Electromagnetics

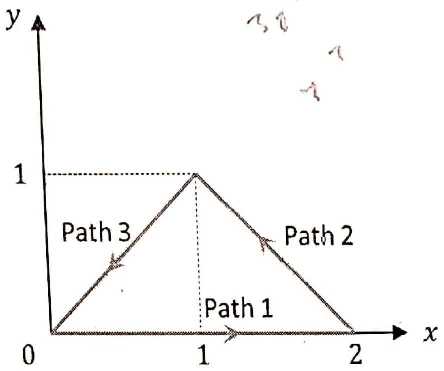
Quiz

1. Vector Calculus

For the vector field $\vec{E} = \hat{x}y^3 + \hat{y}(2x^2 + xy)$, calculate

(a) $\nabla \times \vec{E}$

(b) $\oint_C \vec{E} \cdot d\vec{l}$ around the triangular contour shown below



$(x^2-4x+4)(-x+2)$

$-x^3+4x^2-4x+2x^2-8x+8$

$-x^3+6x^2-12x+8$

$(-y+2)(y)$

$(x^2-4x+4)(-x+2)$

$(-y+2)^2 =$

y^2-8y+8

$+(-y^2+2y)$

y^2-6y+8

$\frac{1}{3}y^3-3y^2+8y$

ⓐ $$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & 2x^2+xy & 0 \end{vmatrix} = \hat{z}\left[\frac{\partial(2x^2+xy)}{\partial x} + \frac{\partial y^3}{\partial y}\right]$$

$$= \boxed{\hat{z}\left[4x+y+3y^2\right]}$$

ⓑ $$\oint_C \vec{E}\cdot d\vec{l} = \cancel{\int_{path①} \vec{E}\cdot d\vec{l}} + \int_{path②} \vec{E}\cdot d\vec{l} + \int_{path③} \vec{E}\cdot d\vec{l}$$

$$= \int_{x=2}^{1} [\hat{x}y^3]\cdot(\hat{x}\,dx)\Big|_{y=-x+2} + \int_{y=0}^{1} [\hat{y}(2x^2+xy)]\cdot(\hat{y}\,dy)\Big|_{x=-y+2}$$

$$+ \int_{x=1}^{0} [\hat{x}y^3]\cdot(\hat{x}\,dx)\Big|_{y=x} + \int_{y=1}^{0} [\hat{y}(2x^2+xy)]\cdot(\hat{y}\,dy)\Big|_{x=y}$$

$$= \left(-\tfrac{1}{4}x^4 + 2x^3 - 6x^2 + 8x\right)\Big|_2^1 + \left(\tfrac{1}{3}y^3 - 3y^2 + 8y\right)\Big|_0^1$$

$$+ \left(\tfrac{1}{4}x^4\right)\Big|_1^0 + \left(y^3\right)\Big|_1^0$$

$$= -0.25 + 5.33 - 0.25 - 1$$

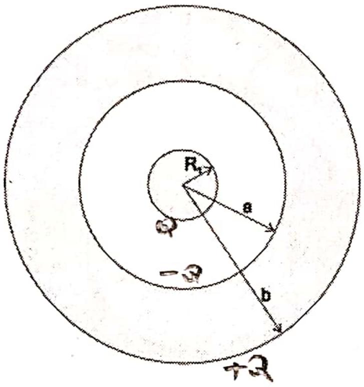
$$= \boxed{3.83}$$

$\frac{1}{3} - 3 + 8$

$-\frac{1}{4}+2-6+8$

2. A metal sphere of radius R_1, carrying charge Q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.
(a) Find the surface charge density ρ_s at R_1, at a, and at b. Make a rough sketch.
(b) Find the E-field in all 4 regions.
(c) Find the potential at the center, using infinity as a reference. Sketch the potential versus R.
(d) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) and (c) change?

-5



(d) → part (a) and (b) do not change (independent of applied grounding) X

→ part (c) changes as the reference is no[w]

$\Rightarrow R = a$

$V = -\int_a^{R_1} \bar{E}\cdot d\bar{l} - \cancel{\int_\infty^a \bar{E}\cdot d\bar{l}}$

$$\boxed{V = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{a}\right)}$$

(a) ρ_s at R_1

$\Rightarrow +Q = \int_s \rho_s \, ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (\rho_s)(R_1^2 \sin\theta \, d\theta \, d\phi)$

$\Rightarrow +Q = \rho_s 4\pi R_1^2$

$\Rightarrow \rho_s = \dfrac{Q}{4\pi R_1^2}$

ρ_s at a → Charge of $-Q$ induced at inner surface of metal shell

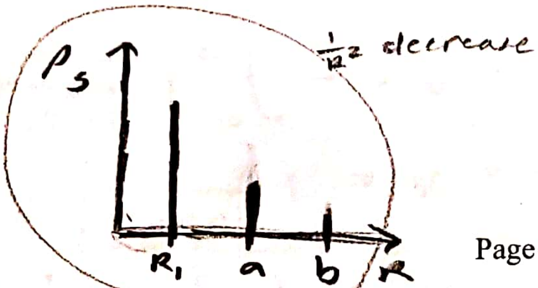
$\Rightarrow -Q = \rho_s 4\pi a^2$

$\Rightarrow \rho_s = \dfrac{-Q}{4\pi a^2}$

ρ_s at b → charge of $+Q$ induced at outer surface of metal shell

$\Rightarrow +Q = \rho_s 4\pi b^2$

$\Rightarrow \rho_s = \dfrac{Q}{4\pi b^2}$



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(b) For $R < R_1$

$\Rightarrow \boxed{\bar{E} = 0}$ (Electric field = 0 inside metal conductor, as all charge is at the surface, $\oint \bar{E}\cdot d\bar{s} = 0$, $\bar{E} = 0$)

For $R_1 \leq R < a$

$\Rightarrow \oint \bar{E}\cdot d\bar{s} = \frac{Q_{enc}}{\varepsilon} = \frac{+Q}{\varepsilon}$, $\bar{E} = E(R)\,\hat{R}$

$\Rightarrow \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (E(R)\cancel{\hat{R}})\cdot(\cancel{\hat{R}}\,R^2 \sin\theta\, d\theta\, d\phi) = -\frac{Q}{\varepsilon}$

$\Rightarrow E(R)\,(4\pi R^2) = +\frac{Q}{\varepsilon}$, $E(R) = \frac{+Q}{4\pi\varepsilon R^2}$

$\Rightarrow \boxed{\bar{E} = \frac{Q}{4\pi\varepsilon R^2}\,\hat{R}}$

For $a \leq R \leq b$

$\Rightarrow \boxed{\bar{E} = 0}$ (E-field = 0 inside metal conductor)

R^{-2}

R^{-1}

$-R$

For $R \geq b$

$\Rightarrow E(R)\,(4\pi R^2) = \frac{Q - Q + Q}{\varepsilon}$, $E(R) = \frac{+Q}{4\pi\varepsilon R^2}$

$\Rightarrow \boxed{\bar{E} = \frac{Q}{4\pi\varepsilon R^2}\,\hat{R}}$

(c) $V = -\int_{\infty}^{0} \bar{E}\cdot d\bar{l} = -\left(\int_{\infty}^{b} \bar{E}\cdot d\bar{l} + \cancel{\int_{b}^{a} \bar{E}\cdot d\bar{l}} + \int_{a}^{R_1} \bar{E}\cdot d\bar{l} + \cancel{\int_{R_1}^{0} \bar{E}\cdot d\bar{l}}\right)$

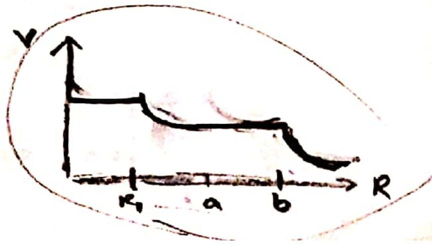
VIP

$\rightarrow V = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\varepsilon R^2}\hat{R}\right)\cdot(\hat{R}\,dR) - \int_{a}^{R_1}\left(\frac{Q}{4\pi\varepsilon R^2}\hat{R}\right)\cdot(\hat{R}\,dR)$

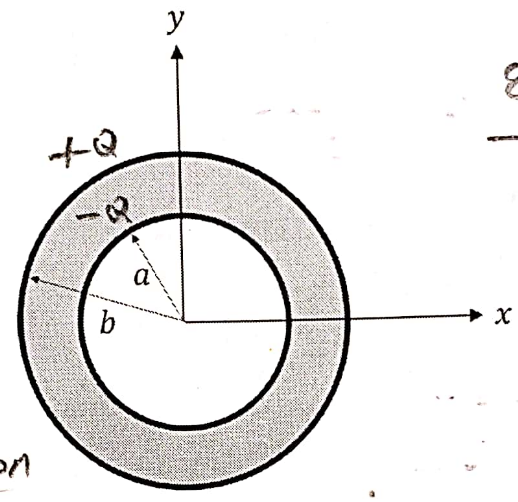
$\rightarrow V = \left(\frac{Q}{4\pi\varepsilon R}\right)\Big|_{\infty}^{b} + \left(\frac{Q}{4\pi\varepsilon R}\right)\Big|_{a}^{R_1}$

(part (d) answered on front of this page)

$\boxed{V = \left(\frac{Q}{4\pi\varepsilon}\left(\frac{1}{b} + \frac{1}{R_1} - \frac{1}{a}\right)\right)}$



3. Two infinitely conducting hollow spheres ($\sigma = \infty$) with infinitely thin surfaces are positioned at the center of the coordinate system. A cross section of the spheres is shown in the figure. The inner sphere has a total charge of Q and the outer one has a charge of $-Q$. The radii of two spheres are $a = 5$ µm, $b = 8$ µm. The region between the two hollow spheres (the grey region) is filled with Barium Titanate having a dielectric constant $\epsilon_r = 1200$ ($\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}$). Find the capacitance of this two-conductor system. (Hint: use spherical coordinates)



$\underline{\varepsilon = \varepsilon_0 \varepsilon_r}$

$\dfrac{1}{\frac{a-b}{ab}}$

$\dfrac{ab}{a-b}$

<u>In dielectric region</u>

$\rightarrow \oint \bar{E} \cdot d\bar{s} = \dfrac{Q_{enc}}{\varepsilon} = \dfrac{-Q}{\varepsilon}$

$\rightarrow \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (E(R)\,\hat{R}) \cdot (\hat{R}\, dR) = -Q/\varepsilon$

$\rightarrow \bar{E} = \dfrac{-Q}{4\pi\varepsilon R^2}\hat{R}$, where $a \leq R < b$

$\rightarrow V = -\int_a^b \bar{E} \cdot d\bar{l} = \cancel{-}\int_a^b \left(\dfrac{\cancel{-}Q}{4\pi\varepsilon R^2}\hat{R}\right) \cdot (\hat{R}\, dR)$

$\rightarrow V = -\dfrac{Q}{4\pi\varepsilon}\left(\dfrac{1}{b} - \dfrac{1}{a}\right)$

$\rightarrow C = \dfrac{Q}{V} = \dfrac{-4\pi\varepsilon}{\left(\frac{1}{b} - \frac{1}{a}\right)} = -4\pi\varepsilon_0\varepsilon_r\left(\dfrac{ab}{a-b}\right)$

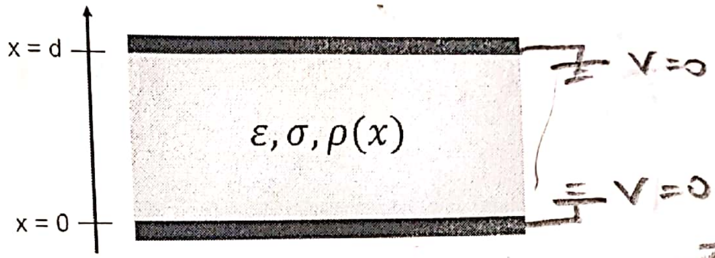
$\rightarrow C = -1.335 \cdot 10^{-7}(-1.33 \cdot 10^{-5})$

$\rightarrow \boxed{C = 1.78 \cdot 10^{-12}\ \mathrm{F}}$

4. For a parallel plate capacitor with dielectric permittivity ε, conductivity σ, and thickness d, assume an initial condition of $\rho(x) = \rho_0$. The two plates were shorted at time $t = 0$ $(V(x=0,\ t=0^+) = V(x=d,\ t=0^+) = 0)$.

(a) Find $V(x, t=0^+)$, $\mathbf{E}(x, t=0^+)$. $\rightarrow$ Apply poisson's Eq's (quasistatic case)

(b) Find $\rho(x, t)$, $V(x, t)$, and $\mathbf{E}(x, t)$.



(a) $\nabla^2 V = -\rho_0/\varepsilon \rightarrow \dfrac{\partial^2 V}{\partial x^2} + \cancel{\dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2}} = -\rho_0/\varepsilon$

$\rightarrow V = -\dfrac{\rho_0}{2\varepsilon}x^2 + Cx + D$

Apply Boundary Condition

$0 = V(x=0) \rightarrow 0x^2 + 0x + D = 0,\ \underline{D=0}$

$0 = V(x=d) \rightarrow 0 = \dfrac{-\rho_0}{2\varepsilon}d^2 + Cd,\ \underline{C = \dfrac{\rho_0}{2\varepsilon}d}$

$\rightarrow \boxed{V(x, t=0^+) = \dfrac{-\rho_0}{2\varepsilon}x^2 + \dfrac{\rho_0}{2\varepsilon}dx}$

$\bar{E} = -\dfrac{\partial V_z}{\partial x}\hat{x}$

$\rightarrow \vec{E} = -\nabla V$ under quasistatic condition, so

$\rightarrow \boxed{\bar{E}(x, t=0^+) = \left(\dfrac{\rho_0}{\varepsilon}x - \dfrac{\rho_0 d}{2\varepsilon}\right)\hat{x}}$

conduction current

(b) $\nabla \cdot \bar{J} = -\dfrac{\partial \rho}{\partial t} \Rightarrow \nabla\cdot(\sigma E) = -\dfrac{\partial \rho}{\partial t}$

$\sigma(\nabla\cdot\bar{E}) = \sigma\left(\dfrac{\rho}{\varepsilon}\right)$

$\rightarrow \sigma\left(\dfrac{\rho}{\varepsilon}\right) = -\dfrac{\partial \rho}{\partial t} \Rightarrow$ Let time constant $\tau = \varepsilon/\sigma$, $\dfrac{\rho}{\tau} = -\dfrac{\partial \rho}{\partial t}$

$\rightarrow \int -\dfrac{1}{\tau}dt = \int \dfrac{1}{\rho}d\rho \rightarrow e^{(-t/\tau)} = e^{\ln(\rho)}$

$\rightarrow \boxed{\rho(x,t) = \rho_0 e^{(-t/\tau)}}$

(Cont. on back)

(part (b) continued)

applying $\rho = \rho_0 e^{(-t/\tau)}$ for quasistatic equation of V and $\bar{E}$ from part (a), we get

$$\boxed{V(x,t) = -\frac{(\rho_0 e^{-t/\tau})}{2\varepsilon} x^2 + \frac{(\rho_0 e^{-t/\tau})}{2\varepsilon} d x}$$

$$\boxed{\bar{E}(x,t) = \left[\frac{(\rho_0 e^{-t/\tau})}{\varepsilon} x - \frac{(\rho_0 e^{-t/\tau}) d}{2\varepsilon}\right] \hat{x}}$$