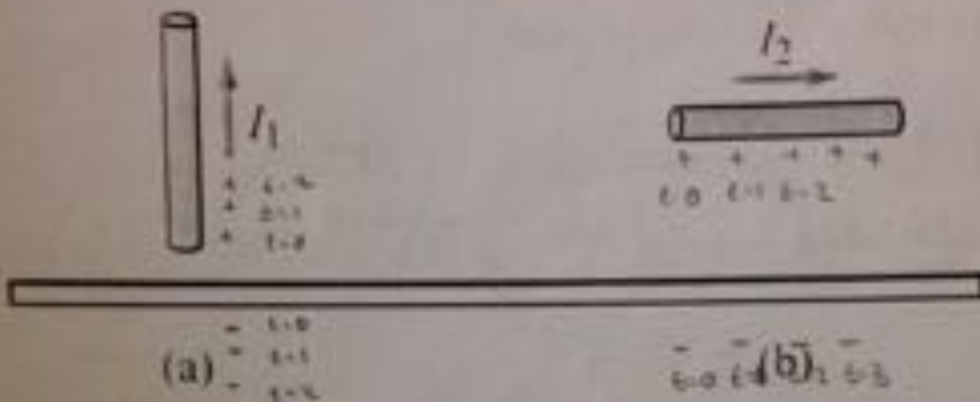
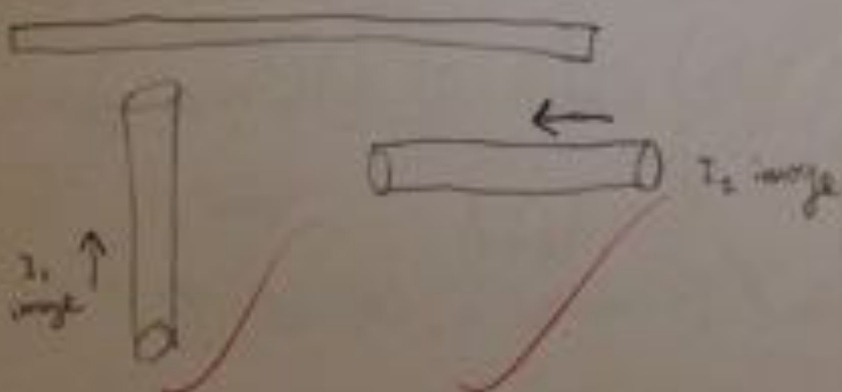
1. Image Theory (10 points)




Conducting wires above a conducting plane carry currents $I_1$ and $I_2$ in the directions shown in the figure above. Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to $I_1$ and $I_2$?




Follow charge at points in time and mirror at every time interval. See original figure for movement of charges.

2. Electrostatics (20 points)

Which of the two following cases does not meet the electrostatic field assumptions? Explain.

(A): $\vec{E} = 4[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

(B): $\vec{E} = 2[y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}]$

$\nabla \times E = 0$ for valid field

$\hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

a) $\hat{x}(0 - 2y) + \hat{y}(0 - 3z) + \hat{z}(0 - x)$

$\boxed{= 4[2y\hat{x} - 3z\hat{y} - x\hat{z}]}$ Not Valid

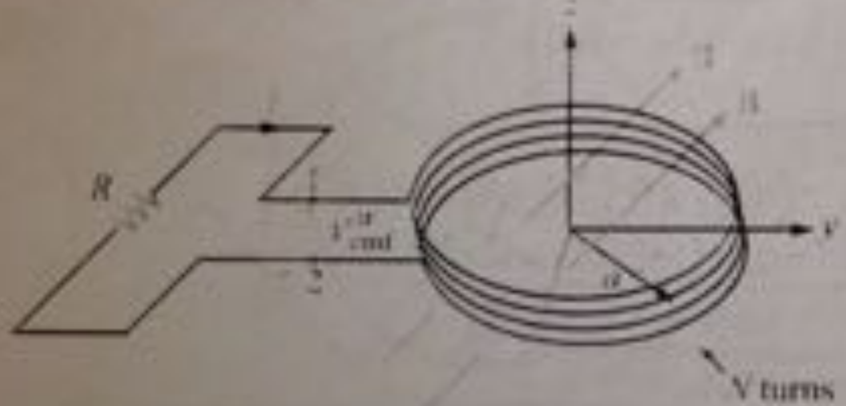
b) $\hat{x}(2z - 2z) + \hat{y}(0 - 0) + \hat{z}(2y - 2y)$

$\boxed{= 0}$ Valid

a is not valid because under static conditions $\nabla \times E = 0$ for conservative field. Field has to be irrotational or curl-free.

EE101A – Engineering Electromagnetics Midterm

3. Inductor in a changing magnetic field (30 points)




$R = \sqrt{x^2 + y^2}$

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the x-y plane with its center at the origin, and connected to a resistor R, as shown in the figure above. In the presence of a magnetic field $\vec{B} = B_0(\hat{y}3 + \hat{z})\sin\omega t$, where $\omega$ is the angular frequency. Note: $\frac{d\sin x}{dx} = \cos x; \frac{d\cos x}{dx} = -\sin x$

Find the following parameters:

$\sin\omega t$

(a) the magnetic flux linking a single turn of the inductor

$$\Phi = \int_s B \cdot ds$$

$$= \iint \left(3B_0\sin(\omega t)\,\hat{y} + B_0\sin(\omega t)\,\hat{z}\right) \cdot \hat{z}\; dx\,dy$$

$$= \iint B_0\sin(\omega t)\; dx\,dy$$

$$\boxed{\Phi = B_0\sin(\omega t)\,\pi a^2}$$

.1m

(b) the $V_{emf}^{tr} = V_1 - V_2$, given that N=10, $B_0$=0.2T, a=10cm, and $\omega=10^3$ rad/s.

Faraday's Law $V_{emf} = -N\frac{\partial \Phi}{\partial t}$

$\Phi = B_0 \sin(\omega t)\pi a^2$

$\frac{\partial \Phi}{\partial t} = B_0 \pi a^2 \omega \cos(\omega t)$

$10cm \cdot \frac{1\ m}{100\ cm}$

$\nabla \times E = \frac{-\partial B}{\partial t}$

$$\boxed{V_{emf} = -N\left(B_0 \pi a^2 \omega \cos(\omega t)\right)}$$

$$= -(10)\left(.2\pi(.1)^2(10^3)\cos(10^3 \cdot t)\right)$$

$$= -2\pi\left((.1)^2(10^3)\cos(10^3 \cdot t)\right)$$

$$\boxed{= -20\pi \cos(10^3 \cdot t)}$$

(c) the polarity of $V_{emf}^{tr}$ at t=0

The polarity of $V_{emf}$ at t=0 is negative. $V_{emf}$ would be $-20\pi$.

(d) the induced current in the circuit for R=1kΩ (assume the wire resistance to be much smaller than R)

$V = IR$ $V = -20\pi\cos(10^3 \cdot t)$

$I = \frac{V}{R}$

$I = \frac{-20\pi\cos(10^3 \cdot t)}{1,000}$ (A)

$$\boxed{I = \frac{-\pi\cos(10^3 \cdot t)}{50} \text{ (A)}}$$

$I = \frac{V_{emf}}{R}$

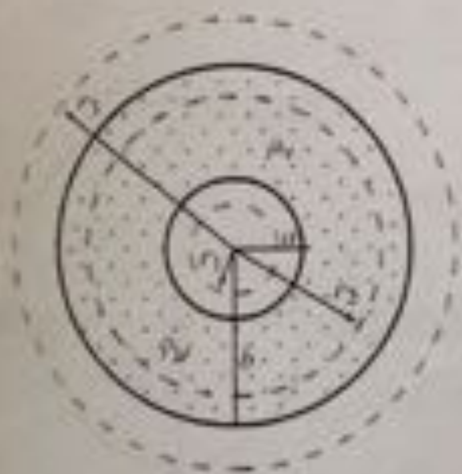
Phasor

$\tilde{V} = -20\pi$

$\tilde{I} = \frac{\tilde{V}}{R} = -\frac{20\pi}{1000} = -\frac{\pi}{50}$

$I = -\frac{\pi}{50}\cos(10^3 \cdot t)$

4. Gauss's Law

$= \cos(\pi) - (-\cos(0))$
$1 - (-1) = 2$

A spherical shell with outer radius b surrounds a charge-free cavity of radius a<b (in the figure above). If the shell contains a charge density given by

$$\rho_v = -\frac{\rho_{v0}}{R^2}, \qquad a \le R \le b,$$

Where $\rho_{v0}$ is a positive constant, determine $\vec{\mathbf{D}}$ in all 3 regions: $a \geq R$, $a \le R \le b$ and $R \ge b$?

$R<a,\ a\le R<b,\ \&\ R\ge b$

$\oint D \cdot ds = Q_{encl}$ $\quad R<a$

$\oint D \cdot ds = 0$

$\vec{D} = \epsilon \vec{E}$ $\quad \boxed{D = 0 \qquad R<a}$ $\quad$ No charge

$\oint D \cdot ds = Q_{encl}$ $\quad a\le R<b$

$\int_0^{2\pi}\int_0^{\pi} D R^2 \sin\theta \, d\theta \, d\phi = Q_{encl}$

$4\pi D R^2 = Q_{encl}$

$4\pi D R^2 = -4\pi \rho_{v0}(R-a)$

$D = \frac{-4\pi \rho_{v0}(R-a)}{4\pi R^2}$

$\boxed{D = -\frac{\rho_{v0}(R-a)}{R^2}\,(\hat{r}) \qquad a\le R<b}$

$Q_{encl} = \int_0^{2\pi}\int_0^{\pi}\int_a^{R} -\frac{\rho_{v0}}{R^2} R^2 \sin\theta \, dR \, d\theta \, d\phi$

$= \int_0^{2\pi}\int_0^{\pi}\int_a^{R} -\rho_{v0} \sin\theta \, dR \, d\theta \, d\phi$

$= 2\pi(2)(-\rho_{v0})(R-a)$

$= -4\pi \rho_{v0}(R-a)$

$\boxed{D = \frac{-\rho_{v0}(b-a)}{R^2}\,(\hat{r}) \qquad R\ge b}$

$\oint D \cdot ds = Q_{encl}$

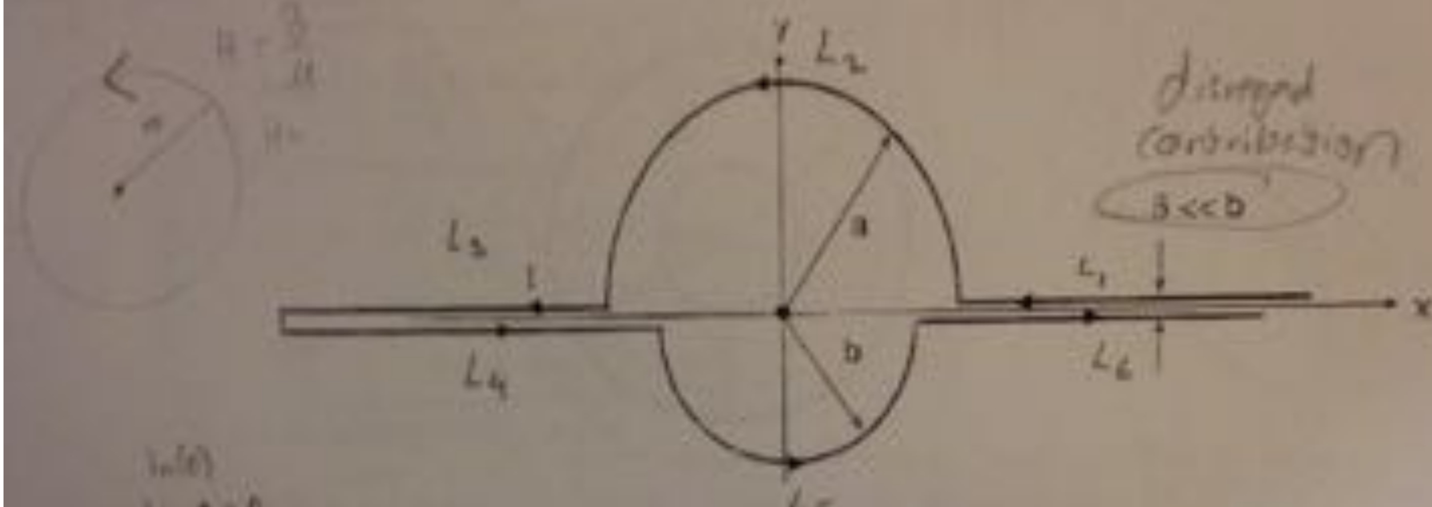
$4\pi D R^2 = Q_{encl}$

$Q_{encl} = \int_0^{2\pi}\int_0^{\pi}\int_a^{b} -\frac{\rho_{v0}}{R^2} R^2 \sin\theta \, dR \, d\theta \, d\phi$

$Q_{encl} = 4\pi(-\rho_{v0})(b-a)$

$D = \frac{Q_{encl}}{4\pi R^2} = \frac{4\pi(-\rho_{v0})(b-a)}{4\pi R^2} = \frac{-\rho_{v0}(b-a)}{R^2}$

5. Bio-Savart Law




The loop shown above consists of two arcs (semi-circles) centered at the origin (x=0, y=0) and the lines parallel to the x-axis as shown below. Determine the magnetic field $\overrightarrow{\mathbf{H}}$ at the origin.

$$H_1 = \frac{I}{4\pi}\int_1 \frac{d\ell \times \hat{R}}{R^2}$$

Since $\delta \ll b$ ... $\hat{\delta}$ for $L_1, L_3, L_4, L_6$ ... $\hat{\ell}$ direction

$$H_3 = \frac{I}{4\pi}\int \frac{d\ell_3 \times \hat{R}}{R^2} \qquad \vec{H} = H_2 + H_5 \qquad \vec{H} = \hat{z} H$$

$$H_6 = \frac{I}{4\pi}\int \frac{d\ell_6 \times \hat{R}}{R^2}$$

$$H_2 = \frac{I}{4\pi}\int_0^{\pi} \frac{(\hat{\phi} \times \hat{R})\, R\, d\phi}{R^2} = \frac{I}{4\pi}\left(\frac{1}{R}\right)\pi = \frac{I\pi}{4\pi R} = \frac{I}{4R}\Big|_{R=a} = \frac{I}{4a}$$

$$H_5 = \frac{I}{4\pi}\int_{\pi}^{2\pi} \frac{R}{R^2}\, d\phi = \frac{I}{4\pi}\left(\frac{1}{R}\right)(2\pi - \pi) = \frac{I\pi}{4\pi R} = \frac{I}{4R}\Big|_{R=b} = \frac{I}{4b}$$

$$\boxed{\vec{H}_{total} = I\left(\frac{1}{4a} + \frac{1}{4b}\right)(\hat{z})}$$