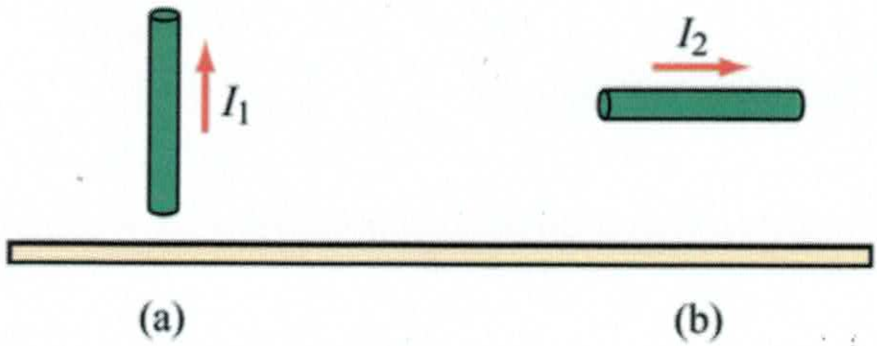
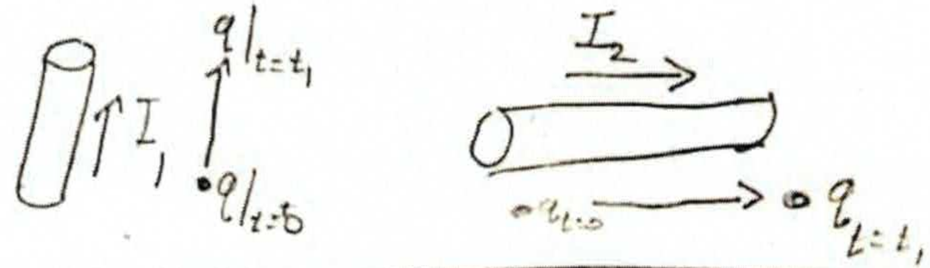
1. Image Theory (10 points)

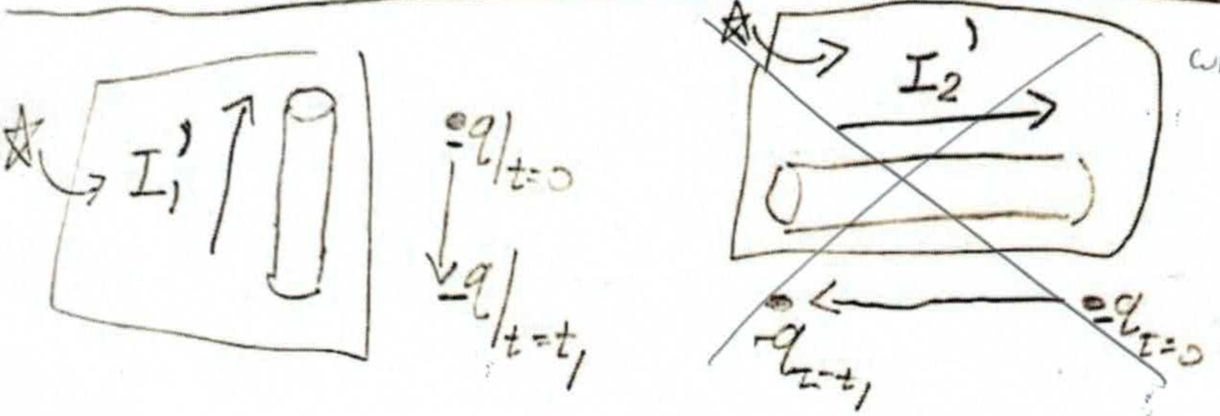
Conducting wires above a conducting plane carry currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the directions shown in the figure above. Keeping in mind that the direction of a current is defined in terms of the movement of positive charges, what are the directions of the image currents corresponding to $\mathbf{I}_1$ and $\mathbf{I}_2$?



ECE 101a Midterm [redacted]

1) a) b)





wrong

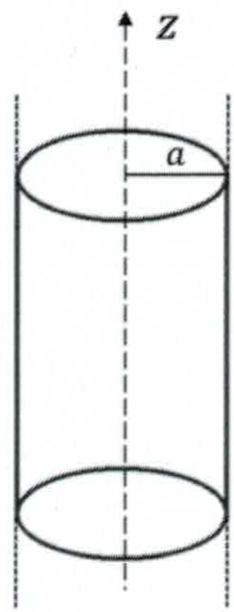
as negative charges flow opposite of conventional current

2. Gauss's Law (15 points)

Consider a long cylinder of radius a with its axis along the z-axis. The cylinder is uniformly charged with a constand *volume* charge density of ρ (positive).

(a) Using Gauss's Law find the electric field **E** as a function of r ($0 \leq r \leq a$ and $r > a$).

(b) Plot **E** as a function of distance r



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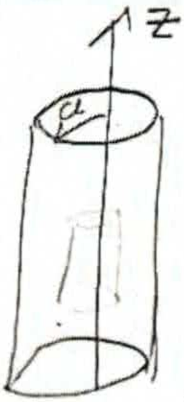
2)



for $0 \le r \le a$

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{Inc}}{\varepsilon_0}$$

$$\oiint \vec{E} \cdot d\vec{s} = \iint E(r)\,\hat{r} \cdot \hat{r}\,ds$$

$$E(r) \cdot 2\pi r \cdot \cancel{h} = \frac{\rho_v \cdot \pi r^2 \cancel{h}}{\varepsilon_0}$$

$$\underline{E(r) = \frac{\rho_v r}{2\varepsilon_0}, \quad \vec{E} = \frac{\rho_v r}{2\varepsilon_0}\hat{r}}$$

for $r > a$

$$\oiint \vec{E} \cdot ds = \frac{Q_{Inc}}{\varepsilon_0}, \quad Q_{inc} = \rho_v \cdot \pi a^2 h$$

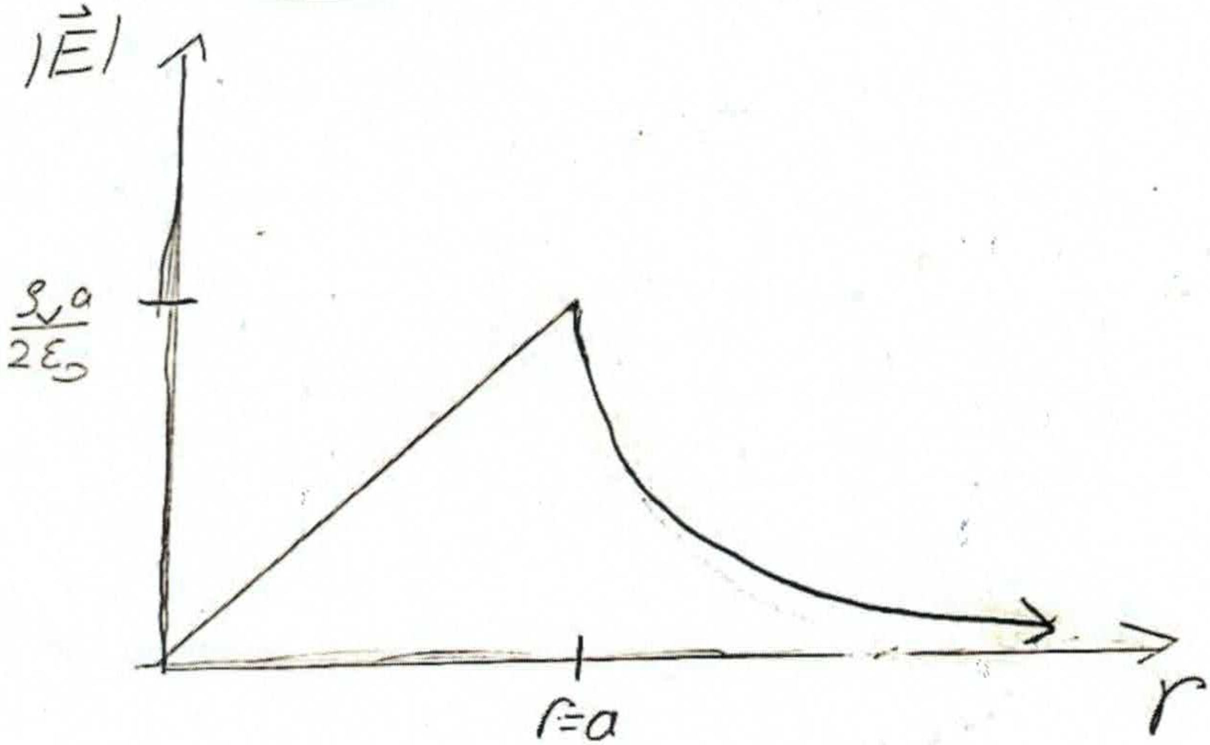
$$E(r) \cdot 2\cancel{\pi} r \cdot \cancel{h} = \frac{\rho_v \cancel{\pi} a^2 \cancel{h}}{\varepsilon_0}$$

$$\underline{E(r) = \frac{\rho_v a^2}{2\varepsilon_0 r}, \quad \vec{E} = \frac{\rho_v a^2}{2\varepsilon_0 r}\hat{r}}$$

part b) on next page
and $\vec{E}$ full notation

$$\boxed{\vec{E} = \begin{cases} \dfrac{\rho_v r}{2\varepsilon_0}\,\hat{r} & \text{for } 0 \le r \le a \\[2ex] \dfrac{\rho_v a^2}{2\varepsilon_0 r}\,\hat{r} & \text{for } r > a \end{cases}}$$

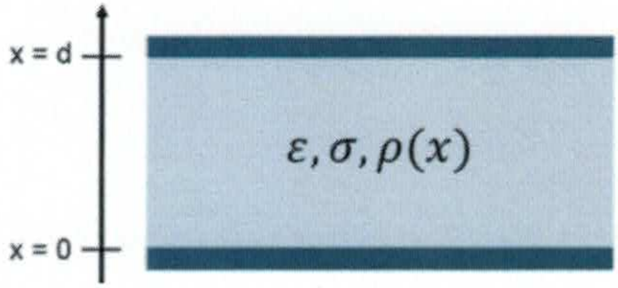
b)



3. Poisson's Equation (15 points)

For a parallel plate capacitor with dielectric permittivity ε, conductivity σ, and thickness d, assume an initial condition of $\rho(x) = \rho_0$. The two plates were shorted at time $t = 0$, so you can assume $V(x=0,\ t=0^+) = V(x=d,\ t=0^+) = 0$.

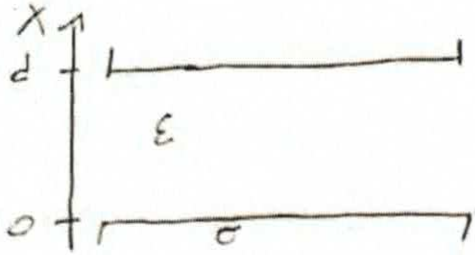
Find $V(x,t=0^+)$, $\boldsymbol{E}(x,t=0^+)$, and surface charge at two plates at $t = 0^+$



3) $\varepsilon, \sigma, d \quad \rho(x) = \rho_0$

shorted @ $t = 0$

Page 1
Q3



$$\frac{d^2V}{dx^2} = -\frac{\rho_0}{\varepsilon}$$

$$\frac{dV}{dx} = \frac{-\rho_0}{\varepsilon} x + C$$

$$V = \frac{-\rho_0}{2\varepsilon} x^2 + Cx + D$$

$$V(x=0) = 0 \rightarrow \quad D = 0$$

$$V(x=d) = 0$$

$$\frac{-\rho_0 d^2}{2\varepsilon} + Cd = 0$$

$$C = \frac{\rho d}{2\varepsilon}$$

$$V(x) = \frac{-\rho_0}{2\varepsilon} x^2 + \frac{\rho_0 d}{2\varepsilon} x$$

next page

$J = \sigma E$, $E = -\frac{dV}{dx}$ $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$

$$\nabla \cdot \vec{J} = \sigma(\nabla \cdot \vec{E})$$

recall maxwell-gauss equation $\nabla \cdot \vec{E} = \rho/\varepsilon$

$$-\frac{d\rho}{dt} = \sigma\left(\frac{\rho}{\varepsilon}\right)$$

$$\frac{d\rho}{dt} + \frac{\sigma}{\varepsilon}\rho = \emptyset$$

$$\frac{d\rho}{dt} \cdot e^{\frac{\sigma}{\varepsilon}t} + \frac{\sigma}{\varepsilon}\rho e^{\frac{\sigma}{\varepsilon}t} = 0$$

$$\frac{d\left[\rho \cdot e^{\frac{\sigma}{\varepsilon}t}\right]}{dt} = 0$$

$$\rho(t)e^{\frac{\sigma}{\varepsilon}t} = C$$

$$\rho(t) = Ce^{-\frac{\sigma}{\varepsilon}t}$$

$$\rho(0) = \rho_0 \Rightarrow C = \rho_0$$

$$\rho(x, t) = \rho_0 e^{-\frac{\sigma}{\varepsilon}t}$$

next page

$$V(x,t) = \frac{-\rho}{2\varepsilon}x^2 + \frac{\rho d}{2\varepsilon}x$$

$$V(x,t) = \left[\frac{-\rho_0 x^2}{2\varepsilon} + \frac{\rho_0 d}{2\varepsilon}x\right] e^{-\frac{\sigma}{\varepsilon}t}$$

$$V(x,t=0^+) = \left[\frac{-\rho_0 x^2}{2\varepsilon} + \frac{\rho_0 d}{2\varepsilon}x\right] \cancel{e^{-\frac{\sigma}{\varepsilon}\cdot 0^+}}\ 1$$

$$\boxed{V(x,t=0^+) = \frac{-\rho_0 x^2}{2\varepsilon} + \frac{\rho_0 d}{2\varepsilon}x}$$

$$\vec{E}(x,t) = -\nabla V(x,t) = -\frac{\partial}{\partial x}\left[\left(-\frac{\rho_0 x^2}{2\varepsilon} + \frac{\rho_0 d}{2\varepsilon}x\right)e^{-\frac{\sigma}{\varepsilon}t}\right]\hat{x}$$

$$\vec{E}(x,t) = -\left[e^{-\frac{\sigma}{\varepsilon}t}\left(\frac{-\rho_0}{\varepsilon}x + \frac{\rho_0 d}{2\varepsilon}\right)\right]\hat{x}$$

$$\vec{E}(x,t) = \left(\frac{\rho_0 x}{\varepsilon} - \frac{\rho_0 d}{2\varepsilon}\right)e^{-\frac{\sigma}{\varepsilon}t}\hat{x}$$

$$\boxed{\vec{E}(x,t=0^+) = \left(\frac{\rho_0 x}{\varepsilon} - \frac{\rho_0 d}{2\varepsilon}\right)\hat{x}}$$

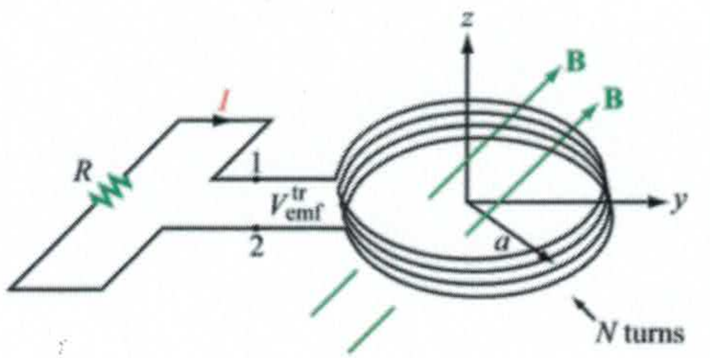
* need $\rho(x, t=0)$

4. Faraday's Law (20 points)

An inductor is formed by winding N turns of thin conducting wire into a circular loop of radius a. The inductor loop is in the x-y plane with its center at the origin, and connected to a resistor R, as shown in the figure below. In the presence of a magnetic field $\vec{B} = B_0(\hat{y} + 3\hat{z})\sin\omega t$, where ω is the angular frequency. Find the following parameters:

(a) the magnetic flux linking a single turn of the inductor

(b) the $V_{emf}^{tr} = V_1 - V_2$, given that $N = 50$, $B_0 = 0.3$ T, $a = 20$ cm, and $\omega = 10^2$ rad/s



4) N turns, radius a, $\vec{B} = B_0(\hat{y} + 3\hat{z})\sin\omega t$

a) $$\Lambda_{\text{single turn}} = N \cdot \Phi \Big|_{N=1}$$

$$\Lambda_{\text{single turn}} = \Phi_B$$

$$\Phi_B = \iint \vec{B} \cdot d\vec{S} = B_z \cdot \pi a^2$$

$$\Phi_B = 3B_0 \sin\omega t \cdot \pi a^2$$

$$\boxed{\Lambda_{\text{single turn}} = 3\pi a^2 B_0 \sin(\omega t)}$$

b) $$V_{emf} = -N \frac{\partial \Phi_B}{\partial t} = V_1 - V_2$$

$$= -N \frac{\partial(3\pi a^2 B_0 \sin\omega t)}{\partial t}$$

$$= -3\pi a^2 B_0 N \omega \cos(\omega t)$$

$$V_{emf} = -3\pi(0.2\,m)^2 \cdot 0.3T \cdot 50 \cdot \frac{100\,rad}{s} \cdot \cos(\omega t)$$

$$\boxed{V_{emf} = -180\pi \cos(100t) \approx 565.49 \cdot \cos(100t)\ V}$$

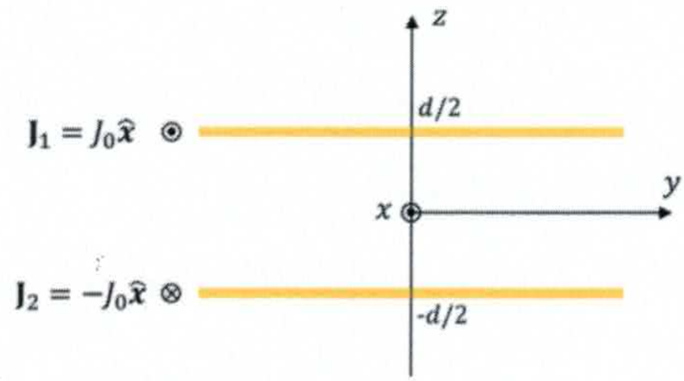
$$-3 \cdot \left(\frac{2}{10}\right)^2 \cdot \frac{3}{10} \cdot 5 \cdot 10 \cdot 10^2 = \frac{-3 \cdot 2^2 \cdot 3 \cdot 5 \cdot 10^3}{10^3} = -180$$

5. Ampere's Law (20 points)

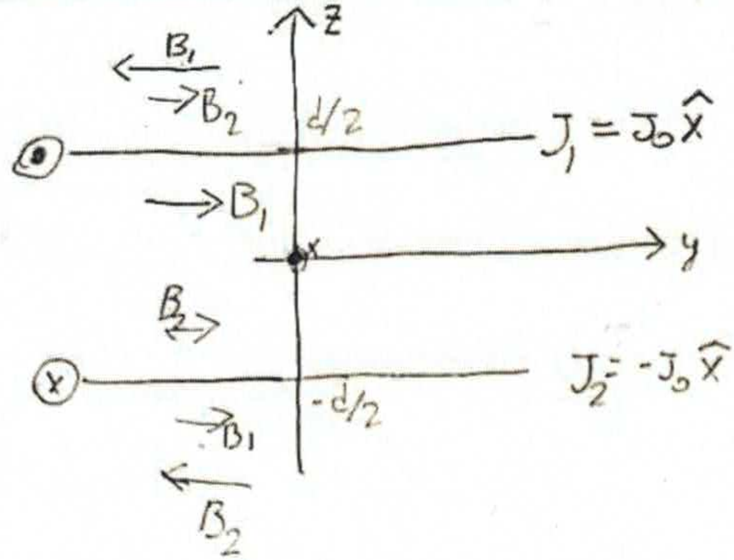
Consider two infinitely large sheets lying in the xy-plane, separated by a distance d. The two sheets carry surface current densities of $\mathbf{J}_1 = J_0\hat{\mathbf{x}}$ and $\mathbf{J}_2 = -J_0\hat{\mathbf{x}}$, respectively (as shown in the figure below). The extent of the sheets in the y direction is infinity. Note that J_0 is the current per unit width perpendicular to the flow.

(a) Find the magnetic field everywhere due to both current sheets.

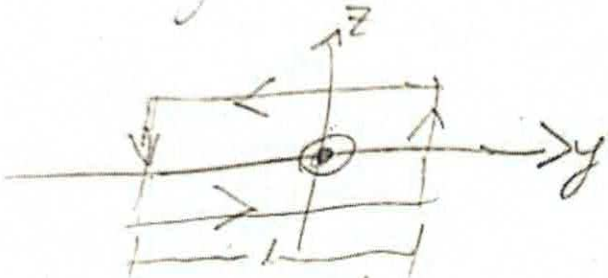
(b) How would your answer in (a) change if both current were running in the same direction, with $\mathbf{J}_1 = \mathbf{J}_2 = J_0\hat{\mathbf{x}}$.



5)



Consider a single infinite sheet of current with $J = J_0 \hat{x}$



$$2Bl = \mu_0 J_0 l$$

$$B = \frac{\mu_0 J}{2} \qquad B = \frac{\mu_0 J_0}{2}(-\hat{y}) \quad z > 0$$

$$= \frac{\mu_0 J_0}{2}\hat{y} \quad z < 0$$

next page

using this result
we see that

$$\vec{B}_1 = \begin{cases} \frac{\mu_0 J_0}{2}(-\hat{y}) & \text{for } z > d/2 \\ \frac{\mu_0 J_0}{2}\hat{y} & \text{for } z < d/2 \end{cases}$$

&

$$\vec{B}_2 = \begin{cases} \frac{\mu_0 J_0}{2}\hat{y} & \text{for } z > -d/2 \\ \frac{\mu_0 J_0}{2}(-\hat{y}) & \text{for } z < -d/2 \end{cases}$$

$$B_{total} = B_1 + B_2 =$$

$$\begin{cases} \cancel{\frac{\mu_0 J_0}{2}(-\hat{y}) + \frac{\mu_0 J_0}{2}(\hat{y})} \to 0 & \text{for } z > d/2 \\ \frac{\mu_0 J_0}{2}(\hat{y}) + \frac{\mu_0 J_0}{2}(\hat{y}) & \text{for } -d/2 < z < d/2 \\ \cancel{\frac{\mu_0 J_0}{2}(\hat{y}) + \frac{\mu_0 J_0}{2}(-\hat{y})} \to 0 & \text{for } z < -d/2 \end{cases}$$

$$\boxed{\vec{B}_{total} = \begin{cases} \mu_0 J_0 \hat{y} & \text{for } -d/2 < z < d/2 \\ \phi & \text{every wher else} \end{cases}}$$

next page for B)

b) now $J_2 = J_1 = J_0 \hat{x}$

B_1 remains unchanged

$$B_2 = B_1' = \begin{cases} \frac{\mu_0 J_0}{2}(-\hat{y}) & \text{for } z > -d/2 \\ \frac{\mu_0 J_0}{2}\hat{y} & \text{for } z < -d/2 \end{cases}$$

$$\vec{B}_{total} = \vec{B}_1 + \vec{B}_2 =$$

$$\begin{cases} \frac{\mu_0 J_0}{2}(-\hat{y}) + \frac{\mu_0 J_0}{2}(-\hat{y}) & \text{for } z > d/2 \\ \cancel{\frac{\mu_0 J_0}{2}\hat{y} + \frac{\mu_0 J_0}{2}(-\hat{y})} = \emptyset & \text{for } -d/2 < z < d/2 \\ \frac{\mu_0 J_0}{2}\hat{y} + \frac{\mu_0 J_0}{2}\hat{y} & \text{for } z < -d/2 \end{cases}$$

$$\boxed{\vec{B}_{total} = \begin{cases} \mu_0 J_0 (-\hat{y}) & \text{for } z > d/2 \\ \emptyset & \text{for } -d/2 < z < d/2 \\ \mu_0 J_0 \hat{y} & \text{for } z < -d/2 \end{cases}}$$

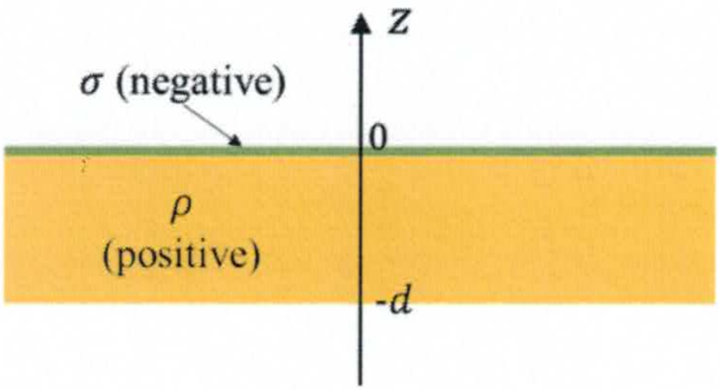
6. Electrostatics (20 points)

As shown in the figure below, an infinite, *non-conducting* sheet (of negligible thickness) carries a negative uniform *surface* charge density σ ($\sigma < 0$). Next to it, an infinite parallel slab of thickness d carries positive uniform *volume* charge density ρ. All charges are fixed.

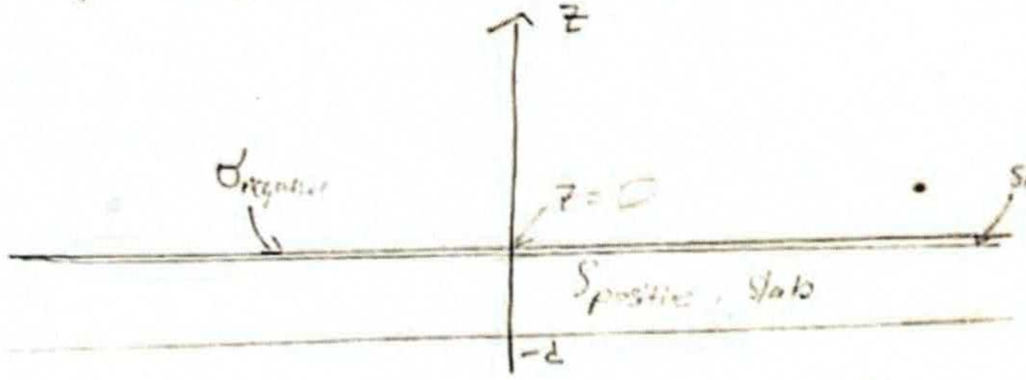
(a) Calculate the magnitude of the electric field **E** in the region above the negatively charged sheet and point out its direction.

(b) Plot **E** as a function of distance z (for $z = 0$ to $+\infty$).

Hint: Gauss's Law and superposition principle could be your best friends when solving this problem! You may want to calculate the electric field due to the sheet of charge alone (let's call it $\mathbf{E}_{\text{sheet}}$), and then calculate the electric field due to the slab of charge alone (let's call it $\mathbf{E}_{\text{slab}}$).



6) $\sigma = \sigma_{sheet} < 0 \qquad \rho_{slab} > 0$



$$\vec{E}_{total} = \vec{E}_{slab} + \vec{E}_{sheet} \quad (\text{Find } \vec{E}|_{z \geq 0}$$

$\vec{E}_{slab}$:

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{Inc}}{\varepsilon_0}$$

$$\oiint E(z)\,\hat{z} \cdot \hat{z}\, ds = \frac{Q_{Inc}}{\varepsilon_0}$$

$$E(z) \cdot \cancel{x \cdot y} = \frac{\rho \cdot \cancel{x \cdot y} \cdot d}{\varepsilon_0}$$

$$\vec{E}_{slab} = \frac{\rho d}{\varepsilon_0}\,\hat{z}$$

E_{sheet}:

$$E(z) \cdot \cancel{x \cdot y} = \frac{\sigma \cdot \cancel{x \cdot y}}{\varepsilon_0}$$

$$\vec{E}_{sheet} = \frac{\sigma}{\varepsilon_0}\,(-\hat{z})$$

next page

$$\vec{E}_{total} = \frac{\rho d}{\varepsilon_0}\hat{z} + \frac{\sigma}{\varepsilon_0}(-\hat{z})$$

$$\boxed{E_1 = \frac{\rho d - \sigma}{\varepsilon_0}\hat{z} \quad \text{for } z > 0}$$

$\boxed{\frac{\rho d + \sigma}{2\varepsilon_0}}$ correct

b)