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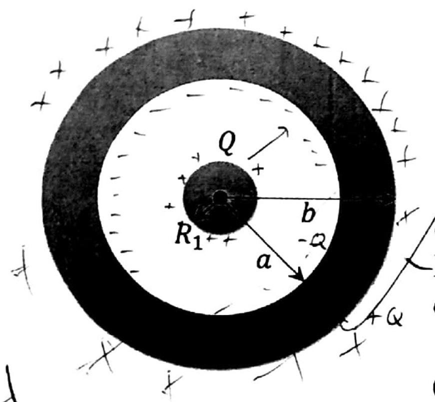
1. Gauss's Law



A metal sphere of radius R_1, carrying charge Q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find the surface charge density ρ_s at R_1, at a, and at b. Make a rough sketch on the figure for the surface charge density at $R = R_1$, $R = a$, and $R = b$.

(b) Find the E-field in all 4 regions.

(c) Find the potential at the center, using infinity as a reference. Sketch the potential versus R.

(a) Since the sphere is perfect → conducting, the surface contains all of Q charge.

$\boxed{\rho_s(R_1) = +\frac{Q}{4\pi R_1^2}}$, $\boxed{\rho_s(a) = -\frac{Q}{4\pi a^2}}$, $\boxed{\rho_s(b) = +\frac{Q}{4\pi a^2}}$

(b) $\vec{E}$ inside conductor is $\vec{0}$. So we only solve for $\vec{E}$ in space ε_0.

$0 \le r < R_1$

$\vec{E} = \vec{0}$

$R_1 < r < a$

Use Gauss's Law $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{s} = \iint_S \rho_s \, ds = Q$.

$\varepsilon_0 \vec{E} \cdot 4\pi r^2 = Q \hat{r}$ $\vec{E}$ must point away from source.

$\boxed{\vec{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}}$

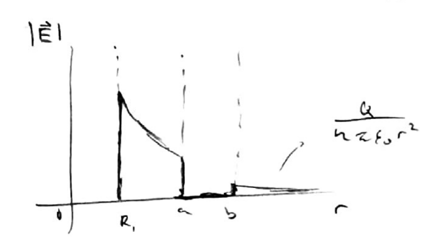
$a < r < b$

$\vec{E}(r) = \vec{0}$ (shown by $Q_{enc} = +Q - Q = 0$, thus $\Phi = 0$ or $E_r = 0$).

$b < r$

$\vec{E}(r) = \frac{(Q - Q + Q)}{4\pi\varepsilon_0 r^2} \hat{r}$

$\boxed{\vec{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \hat{r}}$



(c) $V(\infty) = 0$

conducting objects are equipotential, so $V(0) = V(R_1)$, $V(a) = V(b)$.

$$V(b) - \cancel{V(\infty)}^{0} = -\int_\infty^b \left(\frac{Q}{4\pi\varepsilon_0 r^2} \right) dr$$

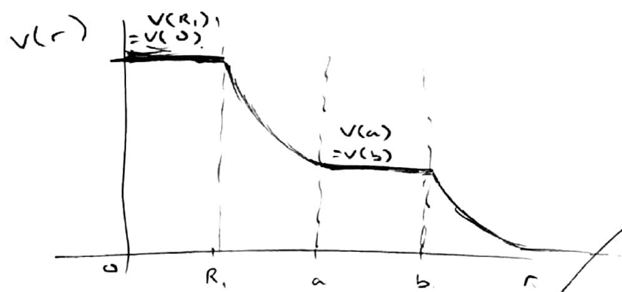
$$V(b) = \frac{Q}{4\pi\varepsilon_0} \left. \frac{1}{r} \right|_\infty^b$$

$$V(b) = \boxed{\frac{Q}{4\pi\varepsilon_0} \frac{1}{b}} = V(a)$$

$$V(R_1) - V(a) = -\int_a^{R_1} \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$V(R_1) - \frac{Q}{4\pi\varepsilon_0} \frac{1}{b} = + \frac{Q}{4\pi\varepsilon_0} \left. \frac{1}{r} \right|_a^{R_1}$$

$$\boxed{V(R_1) = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{a} + \frac{1}{b} \right) = V(0)}$$

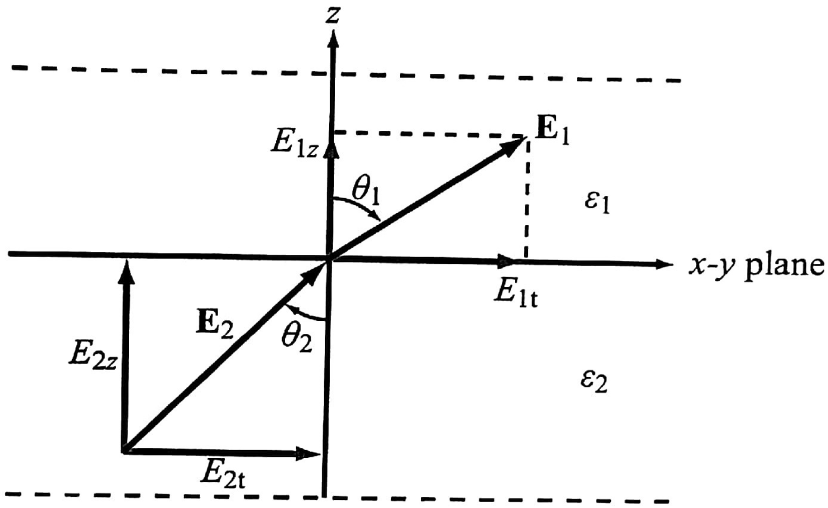


2. Boundary condition

With reference to the figure below,

(a) Find $\vec{E}_1$ if $\vec{E}_2 = \hat{x}3 - \hat{y}2 + \hat{z}2 \left(\frac{V}{m}\right)$, $\varepsilon_1 = 2\varepsilon_0, \varepsilon_2 = 18\varepsilon_0, \rho_s = 8.85 \times 10^{-11} \left(\frac{C}{m^2}\right), \varepsilon_0 = 8.85 \times 10^{-12}\ F/m$.

(b) What angle does $\vec{E}_1$ and $\vec{E}_2$ make with the z axis?



(a) $\vec{E}_{1t} = \vec{E}_{2t}$

$\vec{E}_{1t} = (\hat{x}3 - \hat{y}2)\left(\frac{V}{m}\right)$

$\boxed{\vec{E}_1 = (\hat{x}3 - \hat{y}2 + \hat{z}23)\left(\frac{V}{m}\right)}$

$D_{1z} - D_{2z} = \rho_s$

$D_{1z} = \rho_s + D_{2z} = \rho_s + \varepsilon_2 E_{2z}$

$2\varepsilon_0 E_{1z} = \rho_s + (18\varepsilon_0)(2\ \frac{V}{m})$

$E_{1z} = \left(\frac{46\varepsilon_0}{2\varepsilon_0}\right)\left(\frac{V}{m}\right)$

$E_z = 23\ \frac{V}{m}$

(b) $|\vec{E}_{2t}| = \sqrt{9+4} = \sqrt{13}$

$\theta_2 = \tan^{-1}\left(\frac{E_{2t}}{E_{2z}}\right) = \tan^{-1}\left(\frac{\sqrt{13}}{2}\right)$

$\theta_2 = 60.98^\circ$

$|\vec{E}_{1t}| = \sqrt{9+4} = \sqrt{13}$

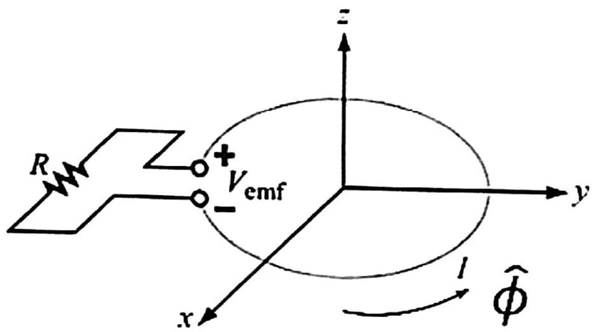
$\theta_1 = \tan^{-1}\left(\frac{\sqrt{13}}{23}\right)$

$\theta_1 = 8.909^\circ$

3. Faraday's Law

The loop in the figure below is in the x-y plane and $\vec{\boldsymbol{B}} = \hat{z}B_0 sin\omega t$ with B_0 positive. What is the direction of I ($\hat{\phi}$ or $-\hat{\phi}$) for the following time points. Please justify your answer:

(a) $t = 0$
(b) $\omega t = \pi/4$
(c) $\omega t = \pi/2$



(a) $-\frac{\partial \Phi}{\partial t} = V_{EMF}$

$\frac{d\vec{B}}{dt} = \hat{z} B_0 \omega \cos \omega t$

$\frac{d\vec{B}}{dt}\Big|_{t=0} = \hat{z} B_0 \omega$ $\qquad -\frac{d\vec{B}}{dt} = -\hat{z} B_0 \omega.$

upward flux is decreasing, so the emf must oppose that change in flux. the induced current is therefore $\boxed{-\hat{\phi}}$, by right-hand rule / curling around $-\hat{z}$.

(b) $\omega t = \frac{\pi}{4}$ Let $\omega = \frac{\pi}{4}$. $t = 1$.

$\frac{d\vec{B}}{dt}\Big|_{t=1} = \hat{z} B_0 \omega \cos\left(\frac{\pi}{4}\right)$ ASSUMING $\omega > 0$ at all times, and $t = [0, \infty)$

$= \hat{z} B_0 \omega \frac{\sqrt{2}}{2}.$

$-\frac{d\vec{B}}{dt} = -\hat{z} B_0 \omega \frac{\sqrt{2}}{2}$, for same reasons as above ($\hat{z}$-dir is + flux).

I is in the $\boxed{-\hat{\phi}}$ direction.

c) $\frac{d\vec{B}}{dt}\Big|_{t=2} = \hat{z} B_0 \omega \cos\left(\frac{\pi}{2}\right)$

$= \vec{0}$

There is $\boxed{\text{no induced current}}$, neither $\hat{\phi}$ or $-\hat{\phi}$

4. Ampere's Law

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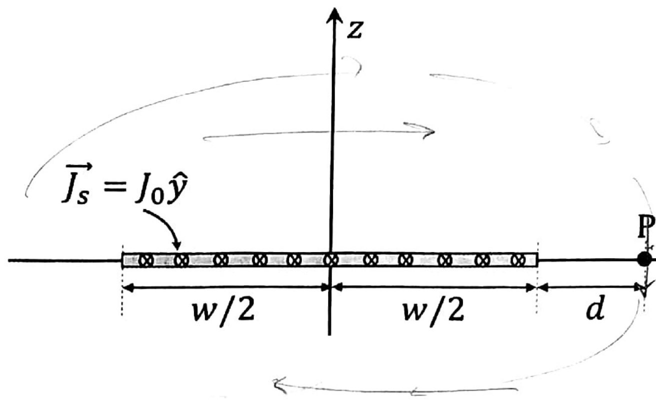

Mag. field must be $\boxed{-\hat{z}}$

1

An infinitely long thin conducting sheet of width w positioned along the x direction and lying in the x-y plane as shown in the figure. It carries a surface current density of $\vec{J_s} = J_0\hat{y}$. Find the magnetic field $\vec{H}$ at point P.

Ampere's Law : $\vec{\nabla} \times \vec{H} = \vec{J}_{enc} + \cancel{\frac{\partial \vec{D}}{\partial t}}^{\to 0}$, static case. $\oint_c \vec{H} \cdot d\vec{l} = I_{enc}$ 5.

$\int_s (\vec{\nabla} \times \vec{H})\, ds = \int_s \vec{J} \cdot d\vec{s}$

$I_{TOT} = J_{s_y} \cdot l = w J_{s_y} = \underline{w J_0}$, $+y$-dir

$\oint \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$

5

Integrate over the bar $\rightarrow$ $-dH_z = \frac{J_0}{2\pi(r+d)}$ 5.

$-H_z = \int_0^w \frac{J_0}{2\pi(r+d)}\, dr$

2

$H_z = -\frac{J_0}{2\pi} \ln(r+d) \Big|_0^w$

$$\boxed{H_z = -\frac{J_0}{2\pi} \ln\left(\frac{w+d}{d}\right)}$$

$$\boxed{\vec{H} = -\hat{z}\, \frac{J_0}{2\pi} \ln\left(\frac{w}{d}+1\right)}$$

2

5. Image Method

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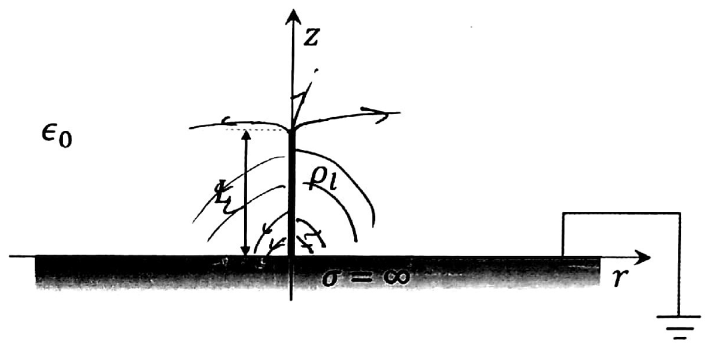


A uniform line charge ρ_l(C/m) of length L stands perpendicularly on a perfectly conducting ground plane of infinite extent in free space with dielectric permittivity ϵ_0.

(a) Find the electric field $\vec{E}(r, z = 0^+)$ at the ground plane surface, where r is the cylindrical radial coordinate shown in the figure above. See integrals in the hint below.

(b) Find the surface charge density on the ground plane surface $\rho_s(r, z = 0^+)$.

Hint for part (a): one or more of the following indefinite integrals may be useful.

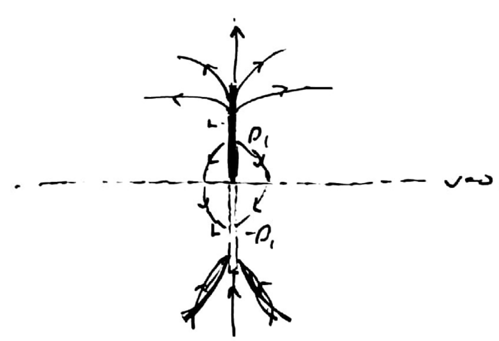
i) $\int \frac{xdx}{(x^2+L^2)^{1/2}} = \sqrt{x^2 + L^2}$

ii) $\int \frac{dx}{(x^2+L^2)^{1/2}} = \ln(x + \sqrt{x^2 + L^2})$

iii) $\int \frac{dx}{(x^2+L^2)^{3/2}} = \frac{x}{L^2(x^2+L^2)^{1/2}}$

iv) $\int \frac{xdx}{(x^2+L^2)^{3/2}} = \frac{1}{(x^2+L^2)^{1/2}}$

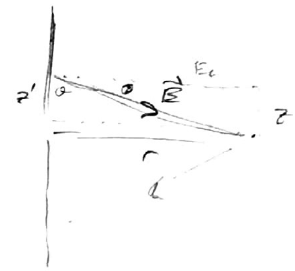
v) $\frac{d\tan\theta}{d\theta} = \frac{1}{\cos^2\theta}$



(a) $\vec{E}(r, z=0) = \vec{E}_+(r, z=0) + \vec{E}_-(r, z=0)$,

that is, the electric field here must be the contributions by both the $+\rho_\ell$ charge rod and its image negative.

$\vec{E}_z = E\cos\theta = E\frac{z'}{R}$



Coulomb's Law

$$\vec{E}_+(r, z=0) = \int_0^L \frac{\rho_\ell \vec{R}}{4\pi\varepsilon_0(r^2+z'^2)^{3/2}} dz' \qquad R = \sqrt{r^2+z'^2}. \qquad \hat{R} = \frac{\vec{R}}{R}$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \int_0^L \frac{1}{(r^2+z'^2)^{3/2}} dz'$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \left. \frac{z'}{r^2(z'^2+r^2)^{1/2}} \right|_{z'=0}^{z'=L} \qquad \text{(iii)}$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \frac{L}{r^2(L^2+r^2)^{1/2}} - 0$$

$$= \frac{\rho_\ell L}{4\pi\varepsilon_0 r^2\sqrt{L^2+0^2}}$$

I crossed this out because...

x-y component negated, only need to calculate z-component!

$$\vec{E}(r, z=0) = \hat{z}2E_{+z}(r, z=0)$$

$dE_z = dE\cos\theta$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0(r^2+z'^2)} \cdot \frac{z'}{\sqrt{(r^2+z'^2)}} dz'$$

$$E_{+z}(r, z=0) = \int_0^L \frac{\rho_\ell z'}{4\pi\varepsilon_0(r^2+z'^2)^{3/2}} dz'$$

$$= \frac{\rho_\ell z'}{4\pi\varepsilon_0} \int_0^L \frac{z'}{(r^2+z'^2)^{3/2}} dz'$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \left. \frac{1}{(z'^2+r^2)^{1/2}} \right|_0^L$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0} \left(\frac{1}{\sqrt{L^2+r^2}} - \frac{1}{r} \right)$$

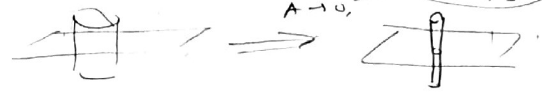
$$\boxed{\vec{E}(r, z=0) = -\hat{z}\frac{\rho_\ell}{2\pi\varepsilon_0}\left(\frac{1}{\sqrt{L^2+r^2}} - \frac{1}{r}\right)}$$

Unit check $\frac{C/m}{F/m}\left(\frac{1}{m}\right) = \frac{V}{m}$ ✓.

(b)

$Q_{enc} = \rho_s A \xrightarrow{A \to 0,} Q_{enc} = \rho_s$

$$\oint E \cdot ds = \frac{Q_{enc}}{\varepsilon_0}$$



$$E_n(r, z=0) = \frac{\rho_s}{\varepsilon_0}$$, minimized Gauss Law w/ $dS \to 0$.

$$E_z(r, z=0) = -\frac{\rho_l}{2\pi\cancel{\varepsilon_0}}\left(\frac{1}{\sqrt{L^2+r^2}} - \frac{1}{r}\right) = \frac{\rho_s}{\cancel{\varepsilon_0}}$$

$$\boxed{\rho_s(r, z=0) = -\frac{\rho_l}{2\pi}\left(\frac{1}{\sqrt{L^2+r^2}} - \frac{1}{r}\right)}$$

$$C/m \left(\frac{1}{m}\right) = \frac{C}{m^2} \checkmark$$