EC ENRG 101A Midterm
Winter 2018

Name: [redacted]
Student ID#: [redacted]

A. Please put all the work on the test paper.
B. Show your work! Provide clear explanations and show calculations. You may not get credit for work that is not shown.

Name of person on LEFT even if "far" way. If wall, then write "Wall". If aisle, then write "Aisle".	ROW NUMBER: (as measured from front)	Name of person on RIGHT even if "far" way. If wall, then write "Wall". If aisle, then write "Aisle".
	10	Aisle

BE SURE TO ENTER THE FOLLOWING INFORMATION

Allowed:
pen/pencil
calculator
formula sheet: ***one side of a*** *8 ½" by 11" sheet of paper.*

Score

1	17 /20		
2	12 /15		
3	30 /30		
4	24 /25		
5	35 /40		
Total	118 /130		

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $d\mathcal{V} =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

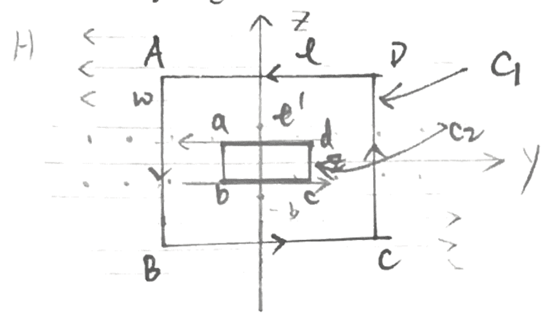
$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Problem 1: (20 points) – Slab of current (hopefully familiar, it's from problem set #3!)

Find the B-field and H-field created by an infinite slab of current density extending in the x-y plane, who density is given by

$$\mathbf{J}(z) = \begin{cases} J_1\hat{x}, & -b < z < b \\ 0, & |z| > b \end{cases}$$

Note #1: J_1 is a constant bulk density with units A/m².
Note #2: Make sure you give the field for all values of z, and don't forget the direction



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As is shown above, the H-field should looks like because of symmetry
And H-fied is constant when $|z|>b$

① when $|z|>b$
make an Amperier countour counter clockwise
Centered at 0, dimension $w \times l$

Amperier's Law

$\oint_{C_1} \vec{H}\cdot d\vec{l} = \int_S J(z)\cdot ds$

$-\cancel{\int_{AB} \vec{H}\cdot d\vec{z}} + \int_{BC} \vec{H}\, d\vec{y} + \cancel{\int_{CD} \vec{H}\, d\vec{z}} - \int_{DA} \vec{H}\cdot d\vec{z}$

$= J_1\hat{x}\cdot 2b\cdot l\cdot \hat{x}$

$\int_{-\frac{l}{2}}^{\frac{l}{2}} H\cdot\hat{y}\,\hat{y}\,dy - \int_{-\frac{l}{2}}^{\frac{l}{2}} H(z)\hat{z}\,dz = J_1 2bl$

$2lH = J_1 2bl$

$H = J_1 b$

$B = \mu H = \mu b J_1$

② when $0<|z|<b$
similary make an Amper countour inside the slab with dimention $l' \times (2z)$

$\oint_{C_2} \vec{H}\cdot d\vec{l} = \int_S J(z)\, ds$

$-\int_{AB} H(z)\cdot(-\hat{z})\cdot\hat{z}\,dz + \int_{BC} H(z)\cdot\hat{z}\cdot\hat{z}\,dz$

$= J_1\hat{x}\cdot\hat{x}\, l' z$

$2H(z)\cdot l' = J_1 l'(2z)$

$H(z) = zJ_1$

$B(z) = z\mu J_1$

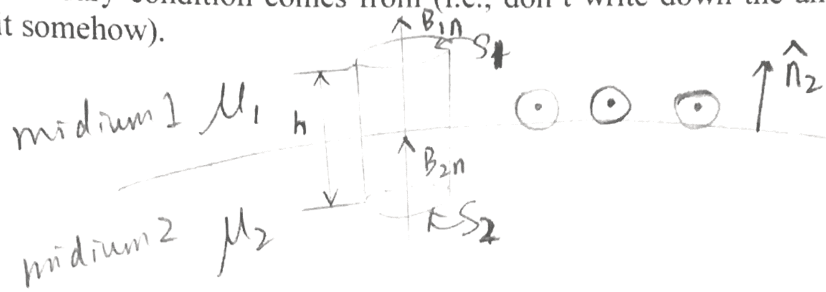
$$\vec{H} = \begin{cases} J_1 b\cdot(-\hat{y}), & z>b \\ z\cdot J_1(-\hat{y}), & 0<z<b \\ z\, J_1(\hat{y}), & -b<z<0 \\ J_1 b(\hat{y}), & z<-b \end{cases}$$

need negative sign or abs value

$$\vec{B} = \begin{cases} \mu b J_1\cdot(-\hat{y}), & z>b \\ z\mu J_1(-\hat{y}), & 0<z<b \\ z\mu J_1(\hat{y}), & -b<z<0 \\ \mu b J_1(\hat{y}), & z<-b \end{cases}$$

3,

Problem 2: (15 points) Magnetic field boundary condition
Derive the boundary condition for the normal components of the H-field across the boundary between medium 1 with $\mu = \mu_1$ and medium 2 with $\mu = \mu_2$.
NOTE: Please make sure your derivation is as clear as possible. You probably have the answer on your cheat sheet, so I want to see on the paper that you obviously know where this boundary condition comes from (i.e., don't write down the answer and try to back into it somehow).



Making a Gauss's surface as above.
Assume h is small, i.e. close to 0 so there is no magnetic field cross the side surface of the cylinder.

Gauss's Law for magnetic field

$$\nabla \cdot \vec{B} = 0$$

$$\oint_S \vec{B} \cdot d\vec{s} = 0$$

$d\vec{s}_2 = -\hat{n}_2 \, ds$
$d\vec{s}_1 = \hat{n}_2 \, ds$

$$\int_{S_2} \vec{B}_{2n} \cdot (-\hat{n}_2) \cdot ds + \int_{S_1} \vec{B}_{1n} \cdot (\hat{n}_2) \, ds = 0$$

$$-B_{2n} S_2 + B_{1n} S_1 = 0$$

$S_1 = S_2 = S$

$$-B_{2n} + B_{1n} = 0$$

$$\boxed{B_{1n} = B_{2n}}$$

$$\boxed{H_{1n} = ?}$$

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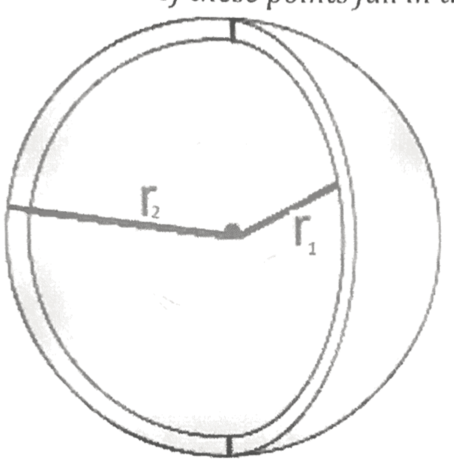
Problem 3: (30 points) Shell of charge

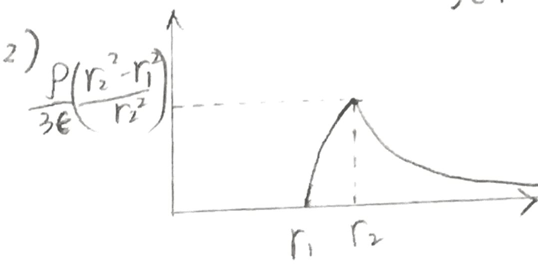
A suspended shell of charge in free space has inner radius r_1 and outer radius r_2. The shell has a uniform charge density, ρ. You may assume there is empty space inside and outside the shell

(1) *Derive an expression for the electric field for all space (as a function of r).*
(2) *Sketch the electric field (as a function of r).*
(3) *What is the voltage between the points (3, 0, -4) and (0, 6, 8)? You may assume both of these points fall in the space outside the outer edge of the shell.*



$$\vec{E}(r) = \frac{D(r)}{\epsilon}\cdot\hat{r} = \begin{cases} 0 & 0<r<r_1 \\ \left(\frac{\rho}{3\epsilon} - \frac{r_1^2\rho}{3\epsilon r^2}\right)\hat{r} & r_1 \le r \le r_2 \\ \frac{(r_2^2-r_1^2)\rho}{3\epsilon r^2}\,\hat{r} & r > r_2 \end{cases}$$

(2)



(1) Make a Gauss's surface that is a sphere concentric with the charged sphere. radius r

$$\oint_s \vec{D}\cdot d\vec{s} = Q_{enclosed}$$

when $0 \le r < r_1$, $Q_{enclosed} = 0$

$$D(r) 4\pi r^2 = 0$$
$$D(r) = 0$$

when $r_1 \le r \le r_2$

$$Q_{enclose} = \int_0^{\pi}\int_0^{2\pi}\int_{r_1}^{r_2} \rho\, r^2 \sin\theta\, dr\, d\phi\, d\theta$$
$$= \frac{4}{3}\pi(r^2 - r_1^2)\rho$$
$$D(r)\cdot 4\pi r^2 = \frac{4}{3}\pi(r^2-r_1^2)\rho$$
$$D(r) = \frac{(r^2-r_1^2)\rho}{3r^2} = \frac{\rho}{3} - \frac{r_1^2\rho}{3r^2}$$

when $r > r_2$

$$Q_{enclose} = \frac{4}{3}\pi(r_2^2 - r_1^2)\rho$$
$$D(r)\, 4\pi r^2 = \frac{(r_2^2-r_1^2)\rho}{3r^2}$$
$$D(r) = \frac{(r_2^2-r_1^2)\rho}{3r^2}$$

(3) When $r > r_2$

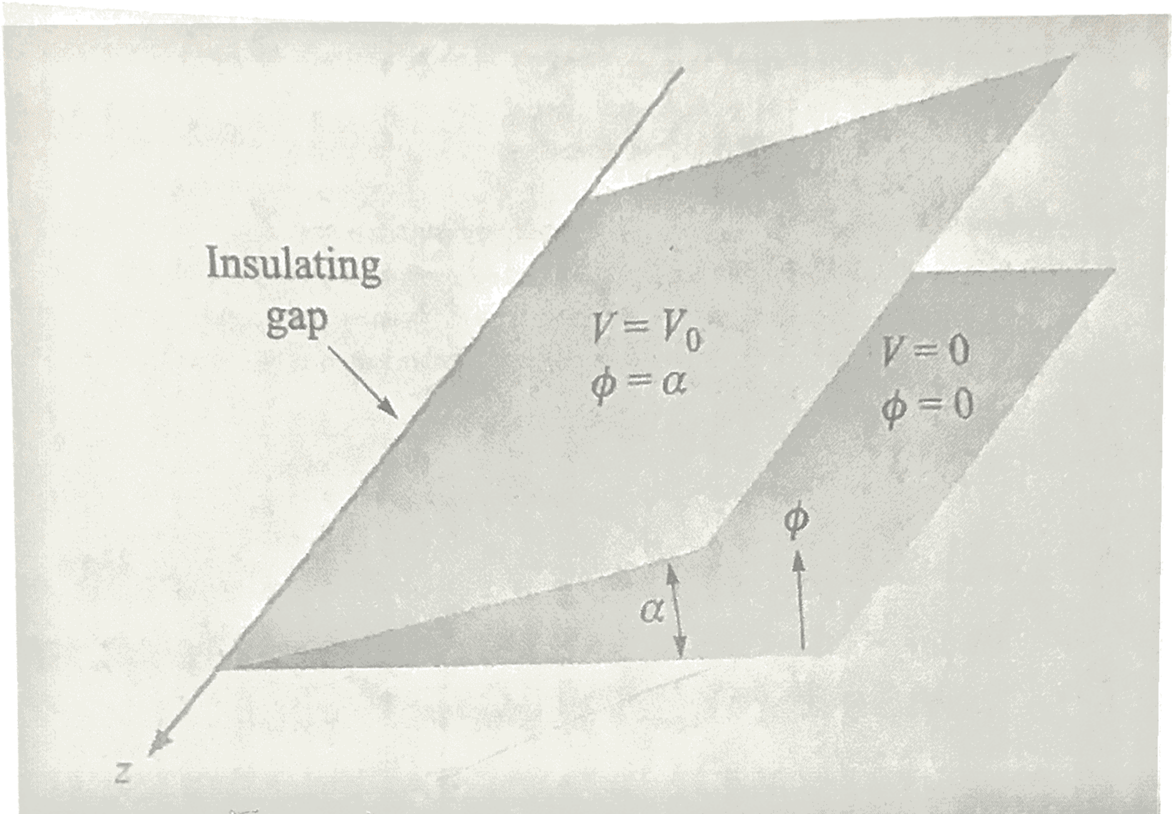
$$V(r) = -\int_{\infty}^{r} \vec{E}(r)\cdot d\vec{l} \qquad d\vec{l} = d\hat{r}$$
$$= \frac{(r_2^2-r_1^2)\rho}{3\epsilon}\int_r^{\infty} -\frac{1}{r^2}\, d\hat{r}$$
$$= \frac{(r_2^2-r_1^2)\rho}{3\epsilon}\left[-\frac{1}{r}\right]_r^{\infty}$$
$$= \frac{(r_2^2-r_1^2)\rho}{3\epsilon r}$$

$(3, 0, -4)$ $R_1 = \sqrt{3^2+0^2+(-4)^2} = 5$

$(0, 6, 8)$ $R_2 = \sqrt{6^2+0^2+8^2} = 10$

$$V = V(R_1) - V(R_2)$$
$$= \frac{(r_2^2-r_1^2)\rho}{3\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$
$$= \frac{(r_2^2-r_1^2)\rho}{3\epsilon}\left(\frac{1}{5} - \frac{1}{10}\right)$$
$$= \frac{(r_2^2-r_1^2)\rho}{30\epsilon}$$

Problem 4: (25 points)
You have planes that extend into infinity in the radial and z directions as shown in the picture below. You know that the first plane, at angle $\phi = 0$, has potential $V = 0$, while the other plane at angle $\phi = \alpha$ has potential $V = V_0$. Derive an expression for the E-field between the two planes.



Given $V = -\int_{\infty}^{r} \vec{E}\, d\vec{l}$

In this case, $V(\phi) = \int_0^{\phi} \vec{E}(r)\cdot d\vec{l} = \int_0^{\phi} \hat{\phi} E(r)\cdot \hat{\phi}\, r\, d\phi$

$d\vec{l} = \hat{\phi}\, r\, d\phi$

$\vec{E}(r) = \hat{\phi}\cdot E(r)$

$= \int_0^{\phi} E(r)\cdot r\, d\phi$

Given $V(\alpha) - V(0) = V_0$

$\int_0^{\alpha} E(r)\cdot r\, d\phi - 0 = V_0$

$E(r)\cdot r \int_0^{\alpha} d\phi = V_0$

$E(r) = \frac{V_0}{\alpha \cdot r}$

$\boxed{\vec{E}(r) = -\frac{V_0}{\alpha \cdot r}\,\hat{\phi}}$

r is the distance to the z axis

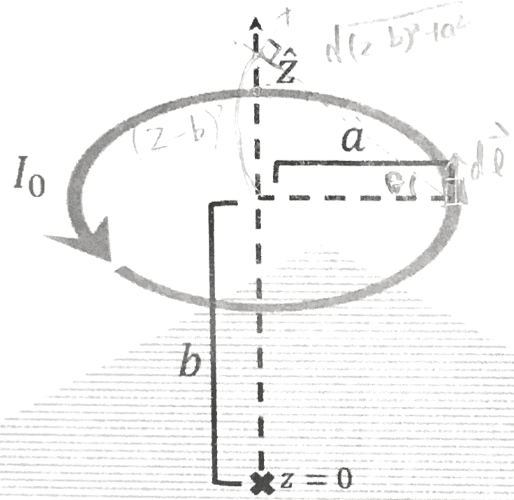
-1

Problem 5 (40 points)

A current loop with radius a, carrying current I_0 (anti-clockwise), is placed at a height $z = b$.

1) Find the magnetic flux density $\vec{B}$ caused by the current loop along the z-axis (through the center of the loop). For this part, you can assume this loop is in free space (i.e., with no conducting plane).
2) Now you can assume there is an infinite, perfectly conducting ground plane at $z = 0$. Where is the image current caused by the current loop in the ground plane? (Hint: to think about an image current, imagine an image charge that you move around.)
3) Which direction is the image current flowing (clockwise or anti-clockwise)?
4) What is the total magnetic flux density $\vec{B}$ along the z-axis from the current loop and ground plane?
5) Where is the magnetic field zero?

(2) at $z = -b$, the direction of current rotation is opposite



(3) opposite of the real current! Anti clockwise

-5

(1) due to symmetry the y.z component of each $I d\vec{\ell}$ contribution is cancelled

$\vec{H}$ is in $-\hat{z}$ direction because of Right hand Rule

Boit - Savart Law

$$d\vec{B} = \frac{I\mu_0}{4\pi}\frac{d\vec{\ell}\times\hat{R}}{R^2}(+\hat{z})$$

$|d\ell \times \hat{R}| = d\ell \quad R = \sqrt{(z-b)^2+a^2}$

$$d\vec{B}_z = \frac{I\mu_0}{4\pi}\frac{d\ell}{(z-b)^2+a^2}(\cos\theta)\hat{z}$$

~~As is shown,~~

$$\cancel{\cos\theta =} \frac{\sqrt{(z-b)^2+a^2}}{a}$$

Take integral

$d\ell = a\,d\phi$

$$\vec{B}(z) = \int_0^{2\pi}\frac{I\mu_0}{4\pi}\frac{d\phi}{\sqrt{(z-b)^2+a^2}}(+\hat{z})$$

$$= \frac{I\mu_0}{2}\frac{1}{\sqrt{(z-b)^2+a^2}}\hat{z}$$

$$\boxed{\vec{B}_1(z) = \frac{I\mu_0}{2\sqrt{(z-b)^2+a^2}}\hat{z}}$$

(4) Similarly to part (1) for the image current generates field in $-z$ direction

$$d\vec{B} = \frac{I\mu_0}{4\pi}\frac{d\vec{\ell}\times\hat{R}}{R^2}\frac{R}{a}(-\hat{z})$$

$|d\vec{\ell}\times\hat{R}| = d\ell = a\,d\phi$

$$\cos\theta = \frac{\sqrt{(z+b)^2+a^2}}{a}$$

$$\vec{B}_2(z) = \frac{I\mu_0}{2}\frac{1}{\sqrt{(z+b)^2+a^2}}(-\hat{z})$$

$$\vec{B}_{tot}(z) = \hat{z}\frac{I\mu_0}{2}\left(\frac{1}{\sqrt{(z-b)^2+a^2}} - \frac{1}{\sqrt{(z+b)^2+a^2}}\right)$$

by superposition principle

(5) $\vec{B} = 0$ at $z = 0$