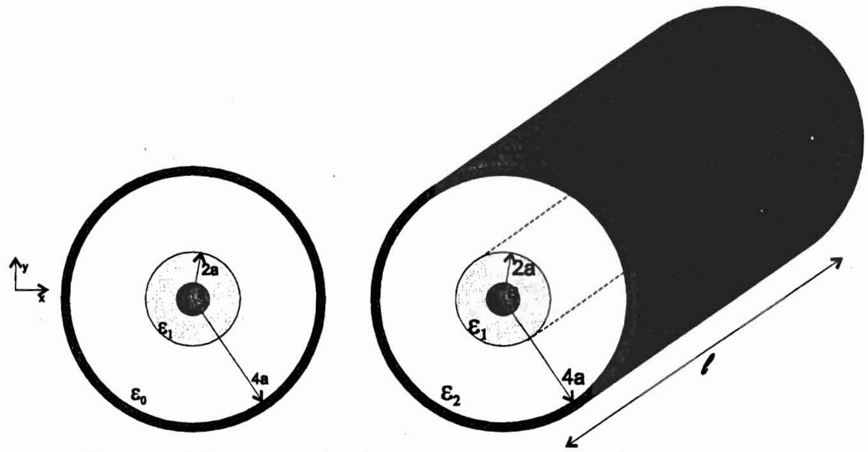
1. Coaxial capacitor (50 points)

Consider a piece of coaxial cable of length ℓ with two dielectric layers with permittivities ε_1 and ε_2. You may consider the inner conductor (radius a) and the outer conductor shell (radius $4a$) to be perfect conductors.



(a) What is the capacitance C between the inner and outer conductors? (Your answer should be in terms of only geometric and material parameters.)

$$E = \frac{\rho_v}{\varepsilon} = \frac{Q}{V\cdot\varepsilon} = \frac{Q}{2\pi r\ell\cdot\varepsilon}$$

$$C = \frac{Q}{V}$$

$$V = -\int E\cdot dl = -\left(\int_a^{2a} E_1\cdot\hat{r}\,dr + \int_{2a}^{4a} E_2\cdot\hat{r}\,dr\right)$$

$$= -\int_a^{2a}\frac{Q\hat{r}}{2\pi\varepsilon_1\ell r}\cdot\hat{r}\,dr - \int_{2a}^{4a}\frac{Q\hat{r}}{2\pi\varepsilon_2\ell r}\cdot\hat{r}\,dr$$

$$= \frac{Q}{2\pi\varepsilon_1\ell}\ln\left(\frac{2a}{a}\right) + \frac{Q}{2\pi\varepsilon_2\ell}\ln\left(\frac{4a}{2a}\right)$$

$$= \frac{Q\cdot\ln(2)}{2\pi\ell}\cdot\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)$$

$$C = \frac{Q}{V} = Q\cdot\frac{2\pi\ell}{\ln(2)Q}\cdot\frac{1}{\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}}$$

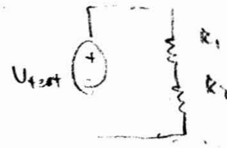
$$= \frac{2\pi\ell}{\ln(2)}\cdot\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}$$ (7)

25

(b) Consider that the dielectric layers are lossy with conductivities σ_1 and σ_2 respectively. What is the resistance R between the inner and outer conductors?

equivalent circuit

$R_{total} = R_1 + R_2$ series resistance



capacitance calculated in part A was total capacitance. if we had taken them individually, ie C_1, C_2 (two capacitors in series):

$$V_1 = \frac{Q \ln 2}{2\pi l \varepsilon_1} \Rightarrow C_1 = \frac{2\pi l \varepsilon_1}{\ln 2}$$

$$V_2 = \frac{Q \ln 2}{2\pi l \varepsilon_2} \Rightarrow C_2 = \frac{2\pi l \varepsilon_2}{\ln 2}$$

correspond to R_1 and R_2, respectively

$$RC = \frac{\varepsilon}{\sigma} \Rightarrow R = \frac{\varepsilon}{C\sigma}$$

$$R_1 = \frac{\varepsilon_1}{C_1 \sigma_1} = \frac{\varepsilon_1 \ln 2}{2\pi l \varepsilon_1 \sigma_1} = \frac{\ln 2}{2\pi l \sigma_1}$$ ✓

similarly, $R_2 = \frac{\ln 2}{2\pi l \sigma_2}$ ✓

$$R_{total} = R_1 + R_2$$

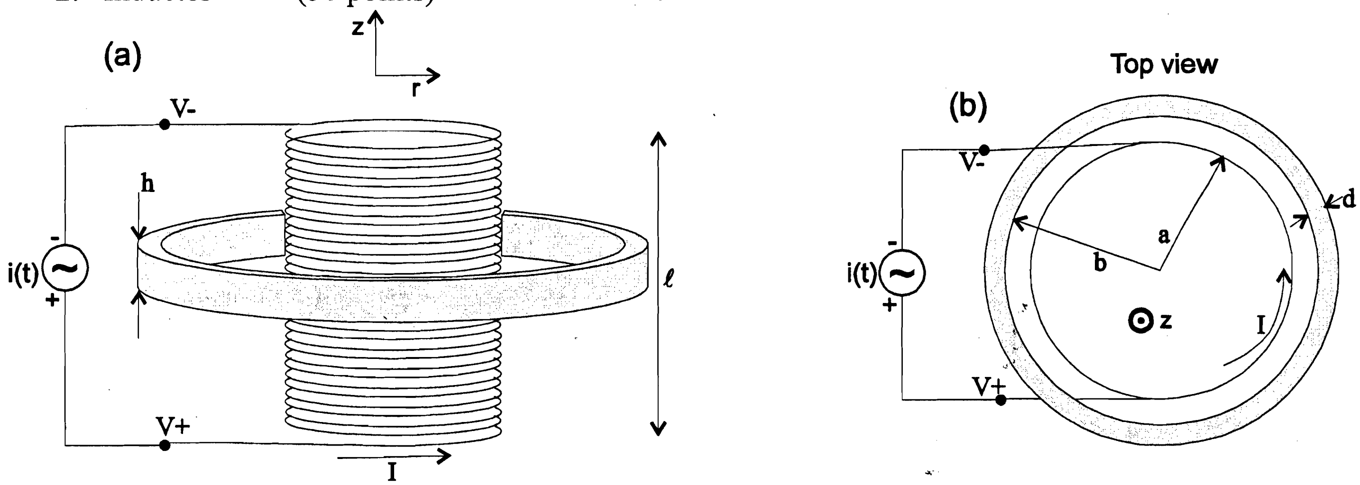
$$= \frac{\ln 2}{2\pi l}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)$$ ✓

$$= \frac{\ln 2}{2\pi l}\left(\frac{\sigma_1 + \sigma_2}{\sigma_1 \sigma_2}\right)$$ ✓ (Ω)

75

28/50

2. Inductor (50 points)



(a) (20 points) For part (a), ignore the metal ring in the figure.

Consider a solenoid with N turns and length ℓ, and radius a, is driven by a current source $i(t)=i_0\cos(\omega t)$. As shown in the figure, we use the convention that a positive current is associated with current in the ϕ direction. You may assume that the solenoid is long ($\ell \gg a$). What is the voltage difference $v(t)=V_+ - V_-$ that appears across the terminals?

Your answer should be in terms of geometric, material, and fundamental parameters only (i.e. N, ℓ, a, b, ϵ, μ, σ, etc.).

$$\vec{H} = H_z \hat{z} = \frac{I \cdot N}{\ell} \hat{z}$$

$$= \frac{i_0 \cos(\omega t)}{\ell} \hat{z}$$

$$\Phi = \oint B \cdot dS$$

$$= \int_0^a \int_0^{2\pi} \frac{\mu i_0 \cos(\omega t)}{\ell} \, d\phi \, dr$$

$$= \frac{2\pi a \mu i_0 \cos(\omega t)}{\ell} \quad -2$$

$$V_{emf} = -\frac{\partial \Phi}{\partial t}$$

$$= -\frac{2\pi a \omega \mu i_0 N^2 \sin(\omega t)}{\ell} \ (V) \quad -5.$$

13/20

(b) (20 points) A metal ring with height h, radius b, and thickness d, and conductivity σ is placed around the center of the solenoid as shown in the figure. You may assume $d \ll b$. What is the current density $\mathbf{J}(t)$ that flows in the metal ring (give vector direction)? When answering this, you may neglect any contribution due the currents $\mathbf{J}$(t) in the metal ring.

according to Faraday's law, the changing magnetic field will induce a current that will in turn create a magnetic field to oppose the first magnetic field

$\therefore$ H created by induced J will be in the direction of $-\hat{z}$, with the magnitude of $\frac{I_0 \cos(\omega t)}{2\pi r}$

$$\Rightarrow H = \frac{I_0 \cos(\omega t)}{2\pi r}(-\hat{z})$$

$$J = \nabla \times H$$

$$= \underbrace{\hat{r}\left(\frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)}_{0} + \hat{\phi}\left(\underbrace{\frac{\partial H_r}{\partial z}}_{0} - \frac{\partial H_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\underbrace{\frac{\partial}{\partial r}(r H_\phi) - \frac{\partial H_r}{\partial \phi}}_{0}\right)$$

$$= -\hat{\phi}\left(\frac{\partial H_z}{\partial r}\right)$$

$$= -\hat{\phi}\,\frac{i_0 \cos(\omega t)}{2\pi r^2} \quad (A/m^2)$$

−10

J has magnitude $\frac{i \cos(\omega t)}{2\pi r^2}$ and flows in the $(-\hat{\phi})$ or clockwise direction

10/20

(c) (~~20~~ 10 points) Does the approximation used in part (b) (neglecting any contribution due the currents **J**(t) in the metal ring) correspond to the low frequency limit (ω is small compared to τ_m^{-1}), or the high frequency limit (ω is large compared to τ_m^{-1})? Give a qualitative explanation why.

5/10

the Magneto Quasi Static approximation only holds when

$$L << \lambda = c/\omega$$

which is to say

$$\omega >> c/L$$

c has dimension of m/s
L has dimension of m
$\therefore$ $\frac{c}{L} \frac{(m/s)}{(m)}$ has dimension 1/s, same as frequency

frequency has to be much greater compared to τ_m^{-1}

physically speaking,

(1) current in solenoid $\xrightarrow{\text{induces}}$ (2) H field in the $\hat{z}$ direction $\xrightarrow{\text{induces}}$ (3) current in the metal ring $\xrightarrow{\text{induces}}$ (4) another H field of its own

if current changes too fast (ω is large), then before the system can get to the (4)th step, current has changed already, which means the current in step (3) will not get a chance to respond. it also means that the H field in step (4) has very little contribution, and that it is reasonable to approximate it as zero

X -5

UCLA Department of Electrical Engineering
EE101 – Engineering Electromagnetics
Winter 2012
Midterm, February 14 2012, (1:45 minutes)

Name ______________________ Student number ______________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit. CIRCLE YOUR FINAL ANSWER.

If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Coaxial Capacitor	50	
Problem 2	Inductor	50	
Problem 3			
Total		100	

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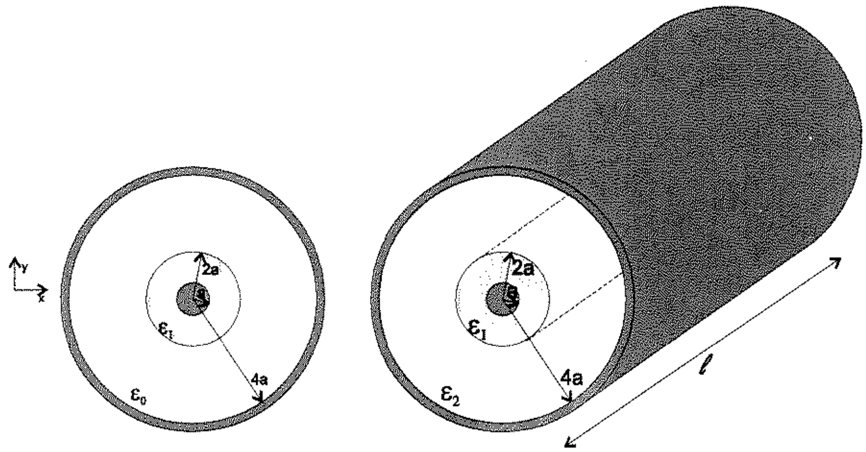
1. Coaxial capacitor (50 points)

Consider a piece of coaxial cable of length ℓ with two dielectric layers with permittivities ε_1 and ε_2. You may consider the inner conductor (radius a) and the outer conductor shell (radius $4a$) to be perfect conductors.



(a) What is the capacitance C between the inner and outer conductors? (Your answer should be in terms of only geometric and material parameters.)

E-field from line charge ρ_ℓ : region 1 $a<r<2a$ $\quad \vec{E}_1 = \hat{r}\dfrac{\rho_\ell}{2\pi r\varepsilon_1}$

region 2 $2a<r<4a$ $\quad \vec{E}_2 = \hat{r}\dfrac{\rho_\ell}{2\pi r\varepsilon_2}$

$$V_1 = V(a) - V(2a) = -\int_{2a}^{a} \frac{\rho_\ell}{2\pi r\varepsilon_1} = \frac{\rho_\ell}{2\pi\varepsilon_1}\ln(2)$$

$$V_2 = V(2a) - V(4a) = -\int_{4a}^{2a} \frac{\rho_\ell}{2\pi r\varepsilon_2} = \frac{\rho_\ell}{2\pi\varepsilon_2}\ln(2)$$

Total potential difference between conductors: $V = V_1 + V_2 = \dfrac{\rho_\ell \ln(2)}{2\pi}\left(\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2}\right)$

Total charge on inner conductor $Q = \rho_\ell \ell$

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\ell}{\ln(2)}\,\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}}$$

Alternate : $C_1 = \dfrac{2\pi\ell\varepsilon_1}{\ln(2)}$ $\quad C_2 = \dfrac{2\pi\ell\varepsilon_2}{\ln(2)}$

$$\boxed{C = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{2\pi\ell}{\ln(2)}\,\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}}$$

(b) Consider that the dielectric layers are lossy with conductivities σ_1 and σ_2 respectively. What is the resistance R between the inner and outer conductors?

For each region $RC = \frac{\varepsilon}{\sigma}$

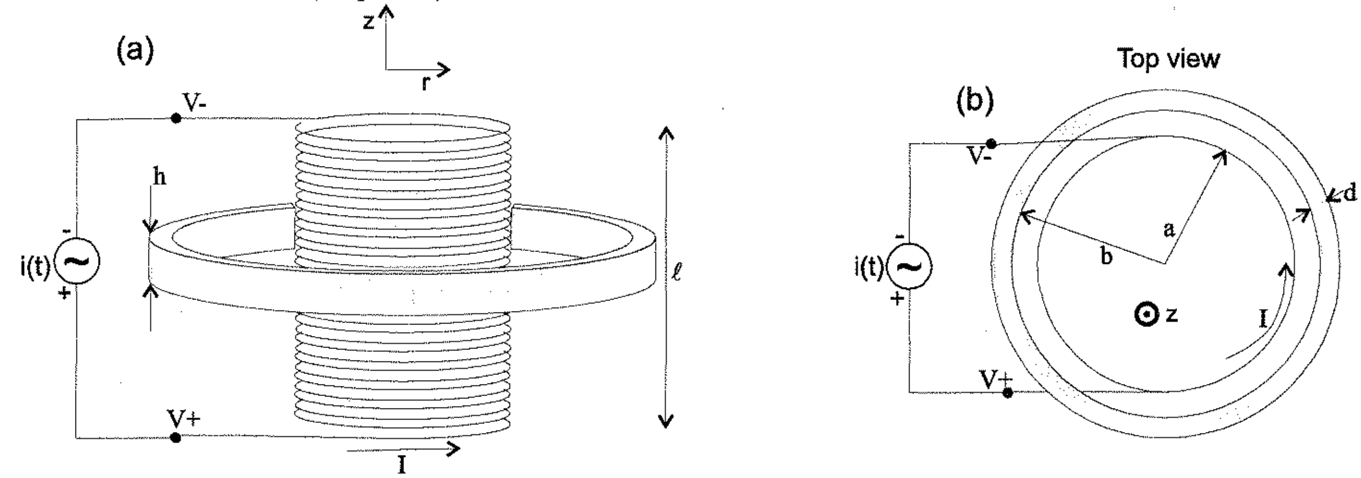
$$R_1 C_1 = \frac{\varepsilon_1}{\sigma_1} \qquad R_2 C_2 = \frac{\varepsilon_2}{\sigma_2}$$

$$R_1 = \frac{\varepsilon_1}{\sigma_1 C_1} \qquad R_2 = \frac{\varepsilon_2}{\sigma_2 C_2}$$

Series Resistors

$$R = R_1 + R_2 = \frac{\ln(2)}{2\pi l}\left(\frac{\sigma_1 + \sigma_2}{\sigma_1 \sigma_2}\right)$$

2. Inductor (50 points)



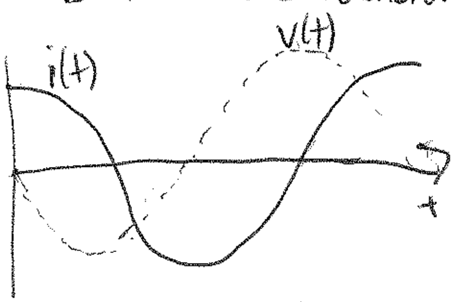
(a) (20 points) For part (a), ignore the metal ring in the figure.

Consider a solenoid with N turns and length ℓ, and radius a, is driven by a current source $i(t)=i_0\cos(\omega t)$. As shown in the figure, we use the convention that a positive current is associated with current in the ϕ direction. You may assume that the solenoid is long ($\ell \gg a$). What is the voltage difference $v(t)=V_+ - V_-$ that appears across the terminals?

Your answer should be in terms of geometric, material, and fundamental parameters only (i.e. N, ℓ, a, b, ϵ, μ, σ, etc.).

Inductance of long solenoid $L = \frac{\mu_0 N^2 A}{\ell} = \frac{\mu_0 N^2 (\pi a^2)}{\ell}$

B-field inside solenoid : $\vec{B}_{inside} = \frac{i(t)\mu_0 N}{\ell}\hat{z}$



$$\Phi = \pi a^2 B(t) = \pi a^2 \frac{\mu_0 N}{\ell} i(t)$$

$$\frac{d\Phi}{dt} = \frac{-i_0 \omega \mu_0 N \pi a^2}{\ell}\sin(\omega t)$$

$$\frac{d\Lambda}{dt} = N\frac{d\Phi}{dt}$$

$$\boxed{V(t) = \frac{-i_0 \omega \mu_0 N^2 \pi a^2}{\ell}\sin(\omega t)}$$

Alternate: Impedance of inductor L is $Z = j\omega L$

Phasor $\frac{\tilde{V}}{\tilde{I}} = Z = j\omega L$ so $\tilde{V} = \tilde{I} j\omega L$

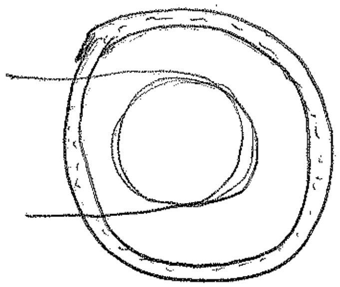
$$V(t) = Re\{i_0 \cos\omega t\, \omega j L\} = -i_0 \omega L \sin(\omega t)$$

$$\boxed{V(t) = -i_0 \omega \frac{\mu_0 N^2 \pi a^2}{\ell}\sin(\omega t)}$$

(b) (20 points) A metal ring with height h, radius b, and thickness d, and conductivity σ is placed around the center of the solenoid as shown in the figure. You may assume $d \ll b$. What is the current density $\mathbf{J}(t)$ that flows in the metal ring (give vector direction)? When answering this, you may neglect any contribution due the currents $\mathbf{J}(t)$ in the metal ring.

Consider Flux inside Metal Loop as a function of time.

From part (a): B-field inside solenoid $B_{inside} = \dfrac{i(t)\mu_0 N}{\ell}\hat{z}$



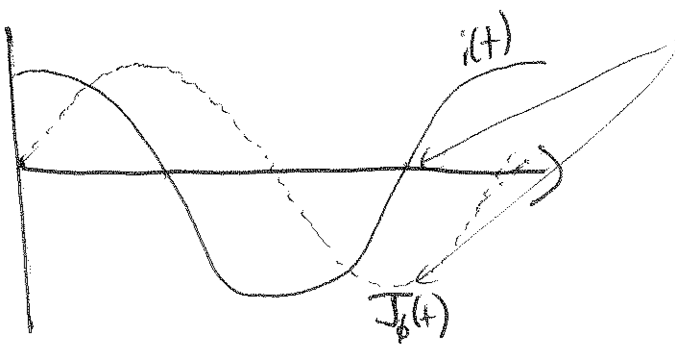
Consider Faraday's Law with loop applied around metal ring.

$$\oint \vec{E}\cdot d\vec{\ell} = \int_0^{2\pi} E_\phi\, b\, d\phi = -\frac{d\Phi}{dt}$$

$$2\pi b E_\phi = +\frac{i_0 \omega \mu_0 N \pi a^2}{\ell}\sin(\omega t)$$

$$E_\phi(t) = +\frac{i_0 \omega \mu_0 N a^2}{2\ell b}\sin(\omega t)$$

$$\boxed{\vec{J} = \hat{\phi}\sigma E_\phi = \frac{i_0 \sigma \omega \mu_0 N a^2}{2\ell b}\sin(\omega t)}$$



When the field inside is increasing most rapidly, J_ϕ is negative & produces clockwise current in metal ring.

If we assume $d \ll b$, then $\vec{J}$ is approximately independent of r! $\vec{J}$ is independent of ϕ by symmetry.

(c) (10 points) Does the approximation used in part (b) (neglecting any contribution due the currents **J**(t) in the metal ring) correspond to the low frequency limit (ω is small compared to τ_m^{-1}), or the high frequency limit (ω is large compared to τ_m^{-1})? Give a qualitative explanation why.

In the low frequency limit, we can neglect the field produced by the currents in the ring. This contribution is proportional to the time derivative of the flux, which increases linearly with ω (Faraday's Law).

$\nabla \times E = -\frac{\partial B}{\partial t} \iff \nabla \times \tilde{E} = -j\omega \tilde{B}$

For $\omega << \tau_m^{-1}$ $\frac{d\Phi}{dt}$ is sufficiently small that the B-field produced by $J(t)$ can be ignored.

Alternatively, we can consider Faraday's Law written in phasor notation $\nabla \times \tilde{\vec{E}} = -j\omega \tilde{\vec{B}}$

The currents in the metal are driven by the solenoidal E-field, which are produced according to Faraday's Law, which are proportional to ω.

$$\tilde{\vec{J}} = \sigma \tilde{\vec{E}}_\phi \hat{\phi} \underset{\text{prop to}}{\propto} j\omega \tilde{\vec{B}}$$

As $\omega \Rightarrow 0$, $\tilde{\vec{J}}$ becomes small, and the resulting B-fields will be much smaller.

Maxwell's Equations in media:

$$\nabla\cdot\mathbf{D} = \rho_f$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

Auxillary Fields: $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$

In linear media: $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}$ $\quad \mathbf{D} = \varepsilon\mathbf{E}$; $\mathbf{M} = \chi_m\mathbf{H}$ $\quad \mathbf{B} = \mu\mathbf{H}$

Ohm's law: $\mathbf{J} = \sigma\mathbf{E}$

Electrostatic Scalar Potential: $\mathbf{E} = -\nabla V$ Vector potential: $\mathbf{B} = \nabla\times\mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \dfrac{\partial\mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla\cdot\mathbf{A})\,dV = \oint_S \mathbf{A}\cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \oint_C \mathbf{A}\cdot d\mathbf{l}$

Electric energy density: $W_e = \dfrac{1}{2}\mathbf{E}\cdot\mathbf{D}$ or $W_e = \dfrac{1}{2}\varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \dfrac{1}{2}\mathbf{B}\cdot\mathbf{H}$ or $W_m = \dfrac{1}{2}\mu H^2$ (in linear media)

Joule power dissipation density: $W_P = \mathbf{E}\cdot\mathbf{J}$ or $W_m = \sigma E^2$ (in Ohm's law media)

Poynting Vector: $\mathbf{S} = \mathbf{E}\times\mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \dfrac{1}{2}\mathrm{Re}\{\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^*\}$

Capacitance: $C = \dfrac{Q}{V}$

Inductance: $L = \dfrac{\Lambda}{I} = N\dfrac{\Phi}{I}$

Boundary conditions: $E_{t,2} - E_{t,1} = 0$ $\quad H_{t,1} - H_{t,2} = J_s$; $D_{n,2} - D_{n,1} = \rho_s$ $\quad B_{n,2} - B_{n,1} = 0$

Bound charge: $\rho_{b,v} = -\nabla\cdot\mathbf{P}$ $\quad \rho_{b,s} = \mathbf{P}\cdot\hat{\mathbf{n}}$

Bound current: $\mathbf{J}_{b,v} = \nabla\times\mathbf{M}$ $\quad \mathbf{J}_{b,s} = \mathbf{M}\times\hat{\mathbf{n}}$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, A =	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A, $\lvert A \rvert$ =	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1}$ =	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, A · B =	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, A × B =	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$ =	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}$ =	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing $\mathbf{A}$ and $\mathbf{B}$

$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

$\nabla(U + V) = \nabla U + \nabla V$

$\nabla(UV) = U\nabla V + V\nabla U$

$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$

$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$

$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$

$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$

$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

$\nabla \cdot (\nabla \times \mathbf{A}) = 0$

$\nabla \times \nabla V = 0$

$\nabla \cdot \nabla V = \nabla^2 V$

$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) \, d\mathcal{V} = \oint_S \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses $\mathcal{V}$)

$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)

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