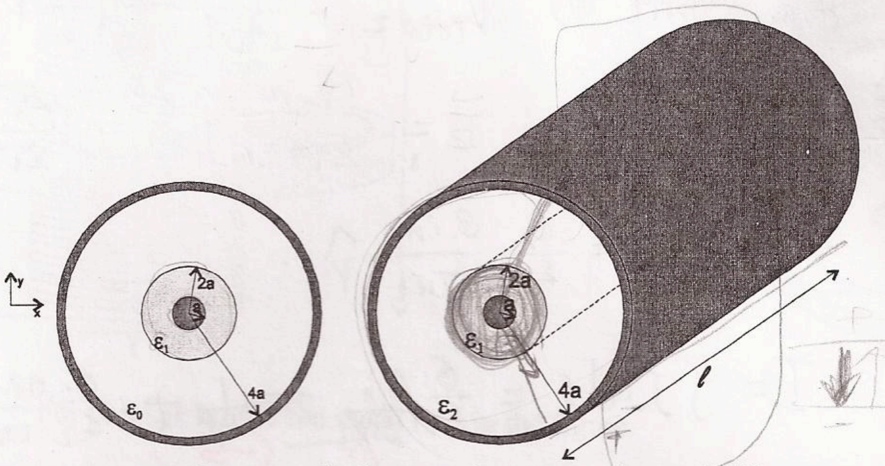
1. Coaxial capacitor (50 points)

Consider a piece of coaxial cable of length $\ell$ with two dielectric layers with permittivities $\varepsilon_1$ and $\varepsilon_2$. You may consider the inner conductor (radius $a$) and the outer conductor shell (radius $4a$) to be perfect conductors.




(a) What is the capacitance $C$ between the inner and outer conductors? (Your answer should be in terms of only geometric and material parameters.)

$$C = \frac{Q}{V} = \frac{\varepsilon \oint E \cdot ds}{-\int E \cdot dl}$$

$E_{N1}\varepsilon_1 + E_{N2}\varepsilon_2$

Using Gauss's Law,

$$\hat{r}\, E\, 2\pi r L = \frac{Q_{inclued}}{\varepsilon}$$

$$\hat{r}\, E = \frac{Q_{in}}{\varepsilon\, 2\pi r L}$$

$$-\frac{Q_{in}}{\varepsilon \pi} \ln\left(\frac{2a}{a}\right) - a$$

when $a < r < 2a$.

$$V = -\int E \cdot dl = -\int_a^{2a} \frac{Q_{in}}{\varepsilon_1 2\pi r L}\, dr = -\frac{Q_{in}}{\varepsilon_1 2\pi L} \ln(r)\Big|_a^{2a} = \frac{Q_{in}}{\varepsilon_1 2\pi L} \ln\left(\frac{1}{2}\right)$$

when $2a < r < 4a$.

$$V = -\int E \cdot dl = -\int_{2a}^{4a} \frac{Q_{in}}{\varepsilon_2 2\pi r L}\, dr = -\frac{Q}{\varepsilon_2 2\pi L} \ln(r)\Big|_{2a}^{4a} = \frac{Q_{in}}{\varepsilon_2 2\pi L} \ln\left(\frac{1}{2}\right)$$

Thus $V_{total} = \frac{Q_{in}}{2\pi L} \ln\left(\frac{1}{2}\right)\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)$ (V)

$$\boxed{C = \frac{Q}{V} = \frac{2\pi L}{\ln(2)\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)} \text{ (F)}}$$

−5

20

(b) Consider that the dielectric layers are lossy with conductivities $\sigma_1$ and $\sigma_2$ respectively. What is the resistance $R$ between the inner and outer conductors?

Since Ohm's Law

$R = \frac{V}{I}$ from part (a) $V_{total} = \frac{Q_{in}\ln(\frac{1}{2})}{2\pi L}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)$ (V)

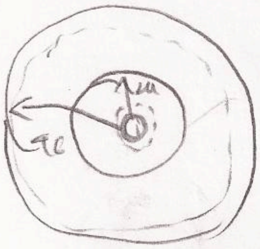
$\vec{J} = \sigma\vec{E}\,\hat{r}$ at $r=a$ $\quad \vec{E} = \frac{Q_{in}}{\varepsilon_1 2\pi r L}\hat{r}$ $\quad \vec{J}_1 = \frac{\sigma_1}{\varepsilon_1}\frac{Q_{in}}{2\pi r L}\hat{r}$

at $r = 4a$. $\quad \vec{J}_2 = \frac{\sigma_2}{\varepsilon_2}\frac{Q_{in}}{2\pi r L}\hat{r}$

Therefore $I = \int J\cdot ds = \frac{\sigma_1}{\varepsilon_1}\frac{Q_{in}}{\cancel{2\pi a L}}\cancel{2\pi a L} + \frac{\sigma_2}{\varepsilon_2}\frac{Q_{in}}{\cancel{2\pi 4a L}}\cancel{2\pi 4a L}$ −5

$= Q_{in}\left(\frac{\sigma_1}{\varepsilon_1}+\frac{\sigma_2}{\varepsilon_2}\right)$ (A)

$\therefore R = \dfrac{\frac{\ln(\frac{1}{2})}{2\pi L}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\right)}{\left(\frac{\sigma_1}{\varepsilon_1}+\frac{\sigma_2}{\varepsilon_2}\right)} = \boxed{\frac{\ln(\frac{1}{2})(\varepsilon_1+\varepsilon_2)}{2\pi L\,(\varepsilon_2\sigma_1+\varepsilon_1\sigma_2)}}$ $\Omega$ −5





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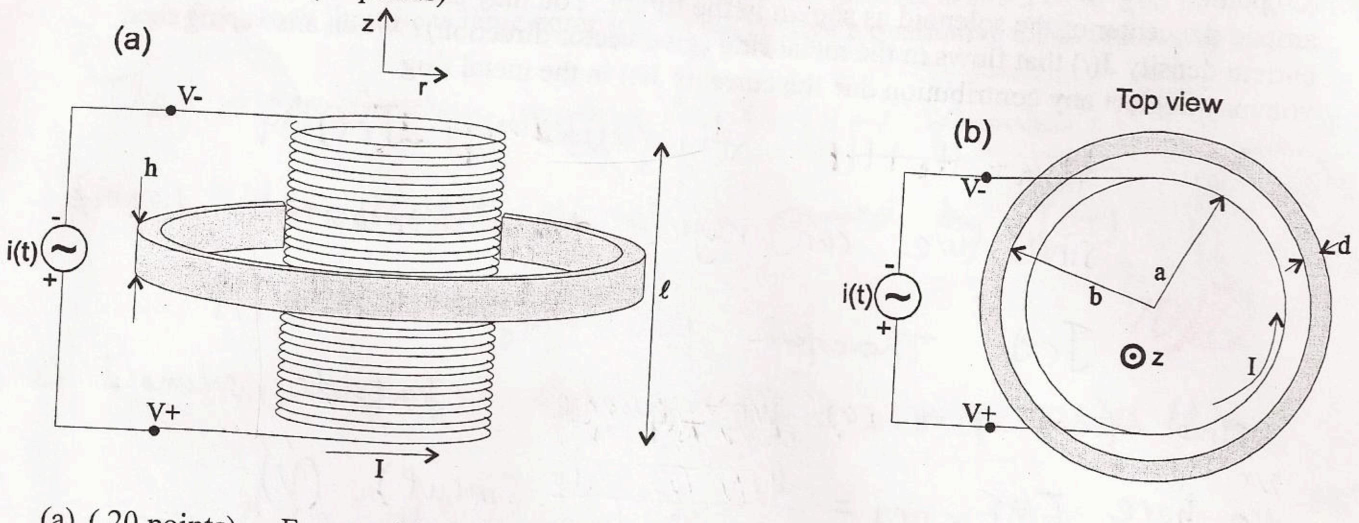
2. Inductor (50 points)




(a) ( 20 points) For part (a), ignore the metal ring in the figure.

Consider a solenoid with $N$ turns and length $\ell$, and radius $a$, is driven by a current source $i(t)=i_0\cos(\omega t)$. As shown in the figure, we use the convention that a positive current is associated with current in the $\phi$ direction. You may assume that the solenoid is long ($\ell \gg a$). What is the voltage difference $v(t)=V_+ - V_-$ that appears across the terminals?

Your answer should be in terms of geometric, material, and fundamental parameters only (i.e. $N$, $\ell$, $a$, $b$, $\epsilon$, $\mu$, $\sigma$, etc.).

Since $EMF = -\frac{d\Lambda}{dt}$. and $i(t) = i_0\cos(\omega t) = \mathrm{Re}\{ \vec{I}_0 e^{j\omega t}\}$.

and for long solenoid, $\Lambda = N\Phi$ = $\vec{I}_0 = i_0$ +10

$\Phi = \int B\cdot ds = \int \frac{\mu_0 i(t) N}{L} ds = \frac{\mu_0 \vec{I} N \pi a^2}{L}$ (assume B outside solenoid $\approx 0$)

$\therefore EMF = -\frac{d(\mu_0 \vec{I} N^2 \pi a^2)}{l\, dt} = -\frac{\mu_0 N^2 \pi a^2}{L}\cdot\frac{d\vec{I}}{dt} = -\frac{\mu_0 N^2 \pi a^2}{L}\mathrm{Re}\{j\omega \vec{I}_0 e^{j\omega t}\}$

$$\boxed{= \frac{\mu_0 N^2 \pi a^2}{L}\, \omega i_0 \sin(\omega t)\ (V).}$$

+8

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(b) (20 points) A metal ring with height $h$, radius $b$, and thickness $d$, and conductivity $\sigma$ is placed around the center of the solenoid as shown in the figure. You may assume $d \ll b$. What is the current density $\mathbf{J}(t)$ that flows in the metal ring (give vector direction)? When answering this, you may neglect any contribution due the currents $\mathbf{J}(t)$ in the metal ring.


$H_{in} = H_0 + H_1$ $H_1$ caused by $J(t)$

since we can neglect the contribution due to $J(t)$, Therefore $H_{in} = H_0$.

since from part (a) we assume $B$ outside solenoid is 0,

we have $\text{EMF}_{\text{on ring}} = \dfrac{\mu_0 N^2 \pi a^2 \omega i_0}{L} \sin(\omega t)$. (V).

$R_{\text{for ring}} = \dfrac{L}{\sigma A} = \dfrac{2\pi b}{\sigma h d}$ and since $d \ll b$.

$\hat{\phi}\, I(t) = \dfrac{\mu_0 N^2 \pi a^2 \omega i_0 \sigma h d}{2\pi b L} \sin(\omega t)$

$= \dfrac{\mu_0 N^2 a^2 i_0 \omega \sigma h d}{L 2 b} \sin(\omega t)$ (A) +10

$\hat{\phi}\, I(t) = \int J(t) \cdot ds = J(t) \cdot h d \cdot \hat{\phi}$

$$\boxed{\therefore J(t) = \hat{\phi}\, \frac{\mu_0 N \cancel{\pi} a^2 \omega i_0 \sigma}{2 b L} \sin(\omega t)\ A/m^2}$$

+8

$E\,dl = 2\pi b. \left( \dfrac{\mu_0 N^2 \pi a^2 \omega i_0 \sigma}{L\, 2\pi b} \right)$.

$DE =$

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(c) (20 points) Does the approximation used in part (b) (neglecting any contribution due the currents **J**(t) in the metal ring) correspond to the low frequency limit ($\omega$ is small compared to $\tau_m^{-1}$), or the high frequency limit ($\omega$ is large compared to $\tau_m^{-1}$)? Give a qualitative explanation why.

The approximation corresponds to $\boxed{\text{low frequen}}$ ✓

Since $H_{in} = H_0 + H_1 \leftarrow$ $H_1$ is caused by $J(t)$.

$H_1 = H_{in} - H_0 = E\sigma d.$ ⇒

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since $\nabla \times E = -\frac{dB}{dt} = -\frac{\mu_0 H_{in}}{dt} \Rightarrow 2\pi r E = \pi r^2 \mu_0 H_{in}$

$\Rightarrow \frac{dH_{in}}{dt} + \frac{H_{in}}{\tau_m} = \frac{H_0}{\tau_m} \qquad \tau_m = \frac{d a \sigma \mu_0}{2}.$

And we assume. $H_{in} = \vec{H}_{in} e^{j\omega t}$. $\quad H_0 = \vec{H}_0 e^{j\omega t}$ plug in.

we have $j\omega H_{in} e^{j\omega t} + \frac{H_{in} e^{j\omega t}}{\tau_m} = \frac{H_0 e^{j\omega t}}{\tau_m}.$

$$H_{in} \cancel{e^{j\omega t}} \left(j\omega + \frac{1}{\tau_m}\right) = \frac{H_0 \cancel{e^{j\omega t}}}{\tau_m}$$

$$\boxed{\frac{H_{in}}{H_0} = \frac{1}{1 + j\omega \tau_m}}$$

So when $\omega \to 0$. $\frac{H_{in}}{H_0} \approx 1$ corresponding to ✓ part(b) since the contribution of $H_1$ is negligable.

When $\omega \to \infty$ $\frac{H_{in}}{H_0} \ll 1$ Thus not correct approximation.