SOLUTIONS

UCLA Department of Electrical Engineering
EE101 – Engineering Electromagnetics
Winter 2009
Midterm, February 9 2008, (1:45 minutes)

Name ______________________ Student number ______________________

This is a closed book exam – you are allowed 1 page of notes (front+back).

Check to make sure your test booklet has all of its pages – both when you receive it and when you turn it in.

Remember – there are several questions, with varying levels of difficulty, be careful not to spend too much time on any one question to the exclusion of all others.

Exam grading: When grading, we focusing on evaluating your level of understanding, based on what you have written out for each problem. For that reason, you should make your work clear, and provide any necessary explanation. In many cases, a correct numerical answer with no explanation will not receive full credit, and a clearly explained solution with an incorrect numerical answer will receive close to full credit.

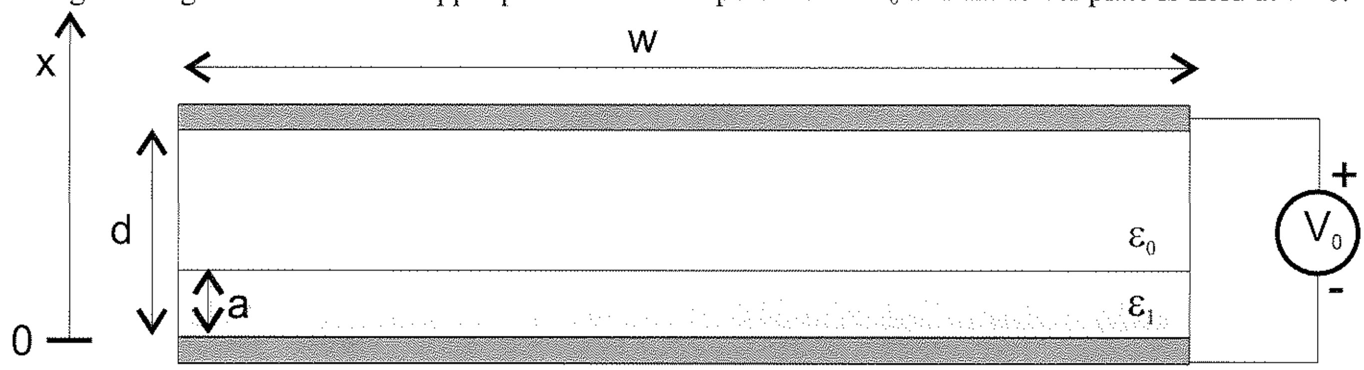
If an answer to a question depends on a result from a previous section that you are unsure of, be sure to write out as much of the solution as you can using symbols before plugging in any numbers, that way at you will still receive the majority of credit for the problem, even if your previous answer was numerically incorrect.

Please be neat – we cannot grade what we cannot decipher.

	Topic	Max Points	Your points
Problem 1	Capacitor	~~40~~ 45	
Problem 2	Magnetic Field	20	
Problem 3	Parallel plate line	~~20~~ 15	
Problem 4	Image theory	20	
Total		100	

1. Capacitor (~~40~~ 45 points)

Consider the parallel plate capacitor shown below, partially filled with dielectric material of permittivity ε_1, The capacitor has depth L (out of the page). There is no free charge in the dielectric (except in part f). Using a voltage source hold the upper plate is held at a potential of V_0 and the lower plate is held at V=0.



(a) (15 points) Find the scalar electrostatic potential $V(x)$ and electric field $\mathbf{E}(x)$ for $0<x<d$.

We consider each region separately and solve LaPlace's Eq $\nabla^2 V = 0 \Leftrightarrow \frac{d^2V}{dx^2}=0$

Region 1: $0<x<a$, ε_1 Region 2: $a<x<d$ ε_0

In each region we consider solutions of LaPlace Eq of the form

$$V_i(x) = A_i x + B_i \qquad i = \text{region index}$$

and satisfy 4 Bnd cond: $V_1(0)=0$, $V_2(d)=V_0$

(for 4 unknowns) $V_1(a)=V_2(a)$, $\varepsilon_1 \frac{dV_1}{dx}\Big|_{x=a} = \varepsilon_2 \frac{dV_1}{dx}\Big|_{x=a}$

$V_1(x=0)=0 \Rightarrow B_1=0 \Rightarrow V_1=A_1 x$

$V_2(x=d)=V_0 \Rightarrow B_2 = V_0 - A_2 d \Rightarrow V_2 = A_2(x-d)+V_0$

1st 2 Bnd cond satisfied.

Apply $V_1(a)=V_2(a) \Rightarrow A_2(a-d)+V_0 = A_1 a \Rightarrow A_1 = A_2\left(1-\frac{d}{a}\right)+\frac{V_0}{a}$

Now we have $V_1 = \left[A_2\left(1-\frac{d}{a}\right)+\frac{V_0}{a}\right]x \qquad V_2 = A_2(x-d)+V_0$ only in terms of A_2

$$\frac{dV_1}{dx} = A_2\left(1-\frac{d}{a}\right)+\frac{V_0}{a} \qquad \frac{dV_2}{dx} = A_2$$

Since εE is continuous at interface $\varepsilon \frac{dV}{dx}$ is continuous - we apply the final

$$\varepsilon_1\left[A_2\left(1-\frac{d}{a}\right)+\frac{V_0}{a}\right] = \varepsilon_0 A_2$$

$$A_2 = \frac{\varepsilon_1 V_0/a}{\varepsilon_0 - \varepsilon_1\left(1-\frac{d}{a}\right)}$$

$$\boxed{V_1(x) = \frac{V_0}{a}\left[\frac{\varepsilon_1\left(1-\frac{d}{a}\right)}{\varepsilon_0-\varepsilon_1\left(1-\frac{d}{a}\right)} + \right]x = \frac{V_0}{a}\left[\frac{\varepsilon_0}{\varepsilon_0-\varepsilon_1\left(1-\frac{d}{a}\right)}\right]x = \left[\frac{V_0}{a+\frac{\varepsilon_1}{\varepsilon_0}(d-a)}\right]x \qquad 0<x<a}$$

$$\boxed{V_2(x) = \frac{V_0}{a}\left[\frac{\varepsilon_1}{\varepsilon_0-\varepsilon_1\left(1-\frac{d}{a}\right)}\right](x-d)+V_0 = \frac{V_0}{a\frac{\varepsilon_0}{\varepsilon_1}+(d-a)}(x-d)+V_0 \qquad a<x<d}$$

$$\boxed{\mathbf{E} = -\frac{dV}{dx} = \begin{cases} -\dfrac{V_0}{a+\frac{\varepsilon_1}{\varepsilon_0}(d-a)}\,\hat{x} & 0<x<a \\[2ex] -\dfrac{V_0}{a\frac{\varepsilon_0}{\varepsilon_1}+(d-a)}\,\hat{x} & a<x<d \end{cases}}$$

in terms of the given Parameters

(b) (5 points) What is the polarization field **P**(*r*) for $0<x<d$? Don't forget the vector direction.

$\vec{P} = \varepsilon_0 \chi_e \vec{E}$ $\qquad \varepsilon_0(1+\chi_e) = \varepsilon_1$ in region 1

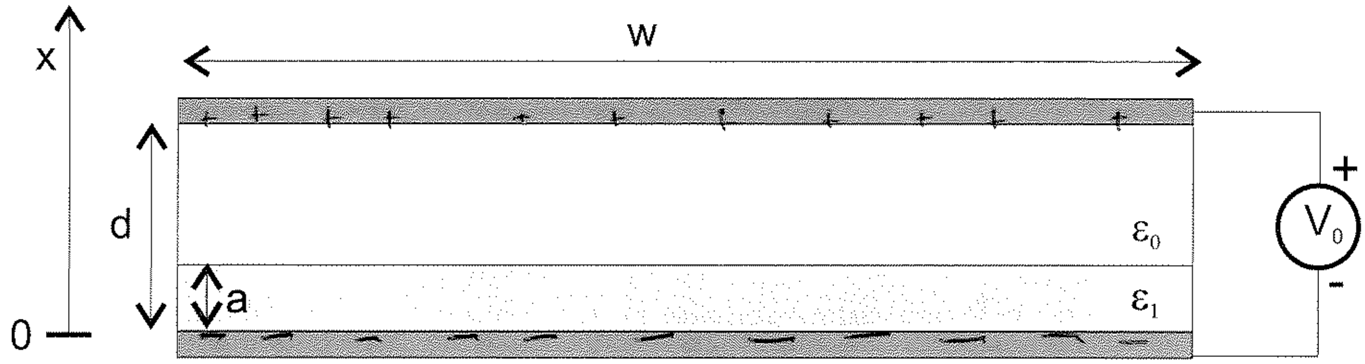
$\left(\chi_e = \frac{\varepsilon_1}{\varepsilon_0} - 1\right)$ $\qquad \chi_e = 0$ in region 2

$$\vec{P} = \begin{cases} -\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)\frac{V_0}{a}\left[\frac{\varepsilon_0}{\varepsilon_0 - \varepsilon_1\left(1 - \frac{d}{a}\right)}\right]\hat{x} & 0<x<a \\ 0 & a<x<d \end{cases}$$

or

$$\vec{P} = \begin{cases} -\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)\frac{V_0}{a + \frac{\varepsilon_1}{\varepsilon_0}(d-a)} & 0<x<a \\ 0 & a<x<d \end{cases}$$

(c) (4 points) Find values for free surface charge density ρ_f on any relevant surfaces, and sketch on the diagram below.



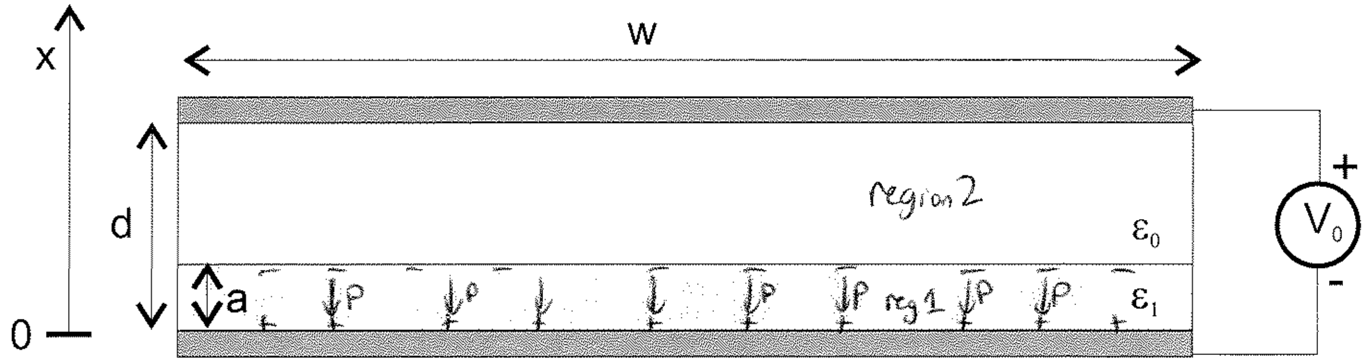
On inner metal surfaces.

$\rho_s = \varepsilon \vec{E} \cdot \hat{n}$

at $x=0$ $\quad \rho_s = -\varepsilon_1 |E| = -\varepsilon_1 \frac{V_0}{a}\left[\frac{\varepsilon_0}{\varepsilon_0 - \varepsilon_1(1-\frac{d}{a})}\right]$

at $x=d$ $\quad \rho_s = \varepsilon_0 |E| = \varepsilon_0 \frac{V_0}{a}\left[\frac{\varepsilon_1}{\varepsilon_0 - \varepsilon_1(1-\frac{d}{a})}\right]$

(d) (5 points) Find values, polarity, and location of any bound charge density ρ_b, and sketch on the diagram below.



Bound charge will accumulate on surface of dielectric.

$\rho_{b,s} = \vec{P} \cdot \hat{n}$

at $x=0$ $\rho_{b,s} = \left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)\frac{V_0}{a}\left[\frac{\varepsilon_0}{\varepsilon_0 - \varepsilon_1(1-\frac{d}{a})}\right]$

at $x=a$ $\rho_{b,s} = -\left[\left(\frac{\varepsilon_1}{\varepsilon_0} - 1\right)\frac{V_0}{a}\left[\frac{\varepsilon_0}{\varepsilon_0 - \varepsilon_1(1-\frac{d}{a})}\right]\right]$

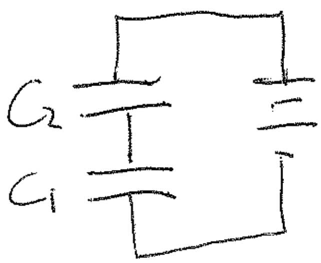
No Bound charge in region 2 since $\varepsilon = \varepsilon_0$

(e) (5 points) What is the capacitance of this structure?

Total charge on plate $= \rho_{s,f} \times W \times L = Q$

$$\boxed{C = \frac{Q}{V_0} = \frac{WL}{a}\left[\frac{\varepsilon_1 \varepsilon_0}{\varepsilon_0 - \varepsilon_1\left(1-\frac{d}{a}\right)}\right]}$$

$$\boxed{C = \frac{A\,\varepsilon_1 \varepsilon_0}{\varepsilon_0 a + \varepsilon_1 (d-a)}}$$

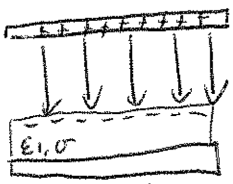


$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_1 = \frac{A\varepsilon_1}{a} \qquad C_2 = \frac{A\varepsilon_0}{d-a}$$

(f) (6 points) For this problem, assume that the dielectric layer is slightly conductive with a conductivity $\sigma = \overset{.001}{1}$ S/m. Assume $\varepsilon_1 = 3\times10^{-11}$ F/m. What is the capacitance at low frequencies, and what is the capacitance at high frequencies? What is the crossover frequency for this behavior? Describe this behavior qualitatively.

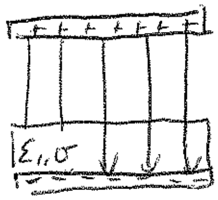
At low freq $\omega \ll \frac{1}{\tau}$ C will be lower as free charge accumulates on the dielectric:



$C \approx C_2$

$$C_{low} = \frac{WL\varepsilon_0}{(d-a)} \quad \text{(smaller)}$$

At high freq $\omega \gg \frac{1}{\tau}$ the free charge won't be able to move across dielectric fast enough. C will be the same as calculated in (e)

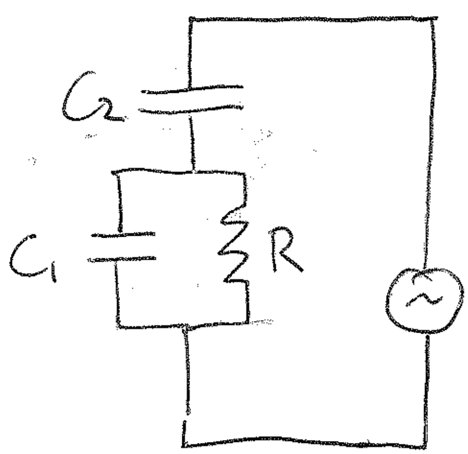


$$C_{high} = \frac{WL}{a} \cdot \frac{\varepsilon_1\varepsilon_0}{\varepsilon_0 - \varepsilon_1\left(1-\frac{d}{a}\right)} \quad \text{(larger)}$$

$$\tau = \frac{\varepsilon_1}{\sigma} = 30 \text{ ns} \quad \text{Dielectric relaxation time}$$

$5.3\ \text{MHz} = \frac{1}{2\pi\tau}$ is the cross-over freq.

Alternate circuit view



$$Z = \frac{1}{j\omega C_2} + \frac{1}{j\omega C_1 + \frac{1}{R}}$$

At high freq $\omega C_1 \gg \frac{1}{R}$

$$Z \approx \frac{1}{j\omega C_2} + \frac{1}{j\omega C_1} \Rightarrow C^{-1} \approx C_1^{-1} + C_2^{-1}$$

C is larger ~~Because~~

At ~~high~~ low freq $\omega C_1 \ll \frac{1}{R}$

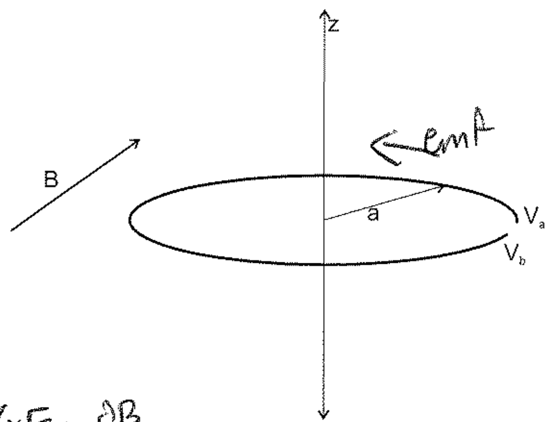
$$Z \approx \frac{1}{j\omega C_2} + R \Rightarrow C \approx C_2$$

only C_2 contributes to capacitance

2. Magnetic Field (20 points)

Consider a loop of wire of radius a, with a very small gap as shown. A static B-field is oriented in the direction $\mathbf{B} = \frac{B_0}{\sqrt{2}}\hat{\mathbf{x}} + \frac{B_0}{\sqrt{2}}\hat{\mathbf{z}}$. The loop is shrinking as a function of time, such that its radius is given by the function $a(t)=a_0-bt$. (for $0 < t < a_0/b$)

(a) What is the voltage across the gap $v(t)=V_b-V_a$ as a function of time for $0 < t < a_0/b$?



If the radius is shrinking, the area is shrinking

$A = \pi a^2 = \pi(a_0-bt)^2$

The Flux is shrinking:

$V_{emf} = 2\pi a(t) E_\phi = -\frac{d\Phi}{dt}$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\int E \cdot dl = -\frac{\partial \Phi}{\partial t}$

$\frac{d\Phi}{dt} = 2\pi(a_0-bt)(-b)\frac{B_0}{\sqrt{2}}$

$$\boxed{V_b - V_a = V_{emf} = 2\pi(a_0-bt)E_\phi = \frac{B_0}{\sqrt{2}} 2\pi b(a_0-bt)}$$

$V_b - V_a =$

(b) Now consider the same problem, except the magnetic field magnitude is changing as a function of time: $B_0(t)=B_0t$, at the same time as the loop is shrinking.

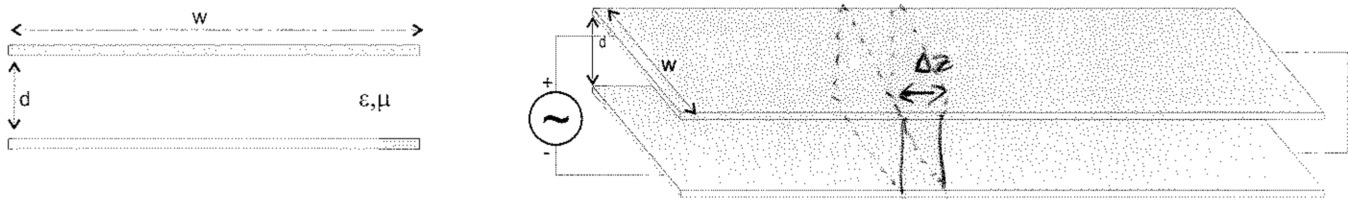
We must consider the total change in flux. $\Phi = \int_S \vec{B}\cdot d\vec{S} = A B_z$

$$\frac{d\Phi}{dt} = \frac{dB(t)}{dt}A(t) + B(t)\frac{dA(t)}{dt}$$

$$\frac{d\Phi}{dt} = \frac{B_0}{\sqrt{2}}\pi(a_0-bt)\left[a_0-3bt\right]$$

$$\boxed{V_b-V_a = -\frac{B_0\pi}{\sqrt{2}}(a_0-bt)\left[a_0-3bt\right]}$$

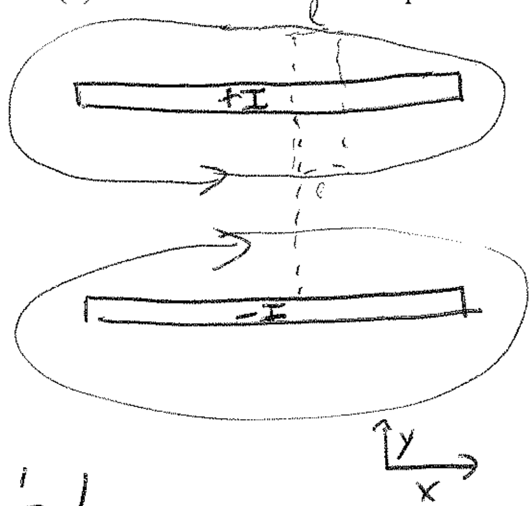
3. Parallel plate transmission lines (20 points)



(a) Consider the parallel plate transmission line shown above composed of two perfectly conducting thin plates. The width is much greater than the plate separation $w \gg d$. What is the capacitance per unit length?

$$C' = \frac{C}{\ell} = \frac{\varepsilon w}{d}$$

(b) What is the inductance per unit length ?



$L' = \frac{L}{\Delta z}$

Between plates B is uniform

$2B_x l = \frac{l I}{W}\mu_0$

$B_x = \frac{I}{2W}\mu_0$ from one plate

$B_x = \frac{I}{W}\mu_0$ in between plates

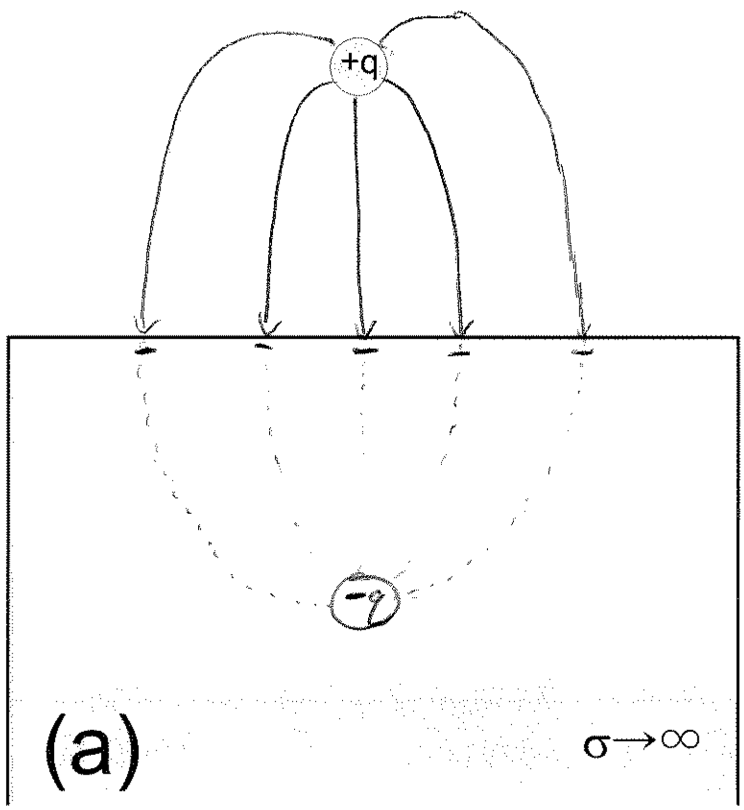
$\Phi = \frac{I\mu}{W}\Delta z\, d$

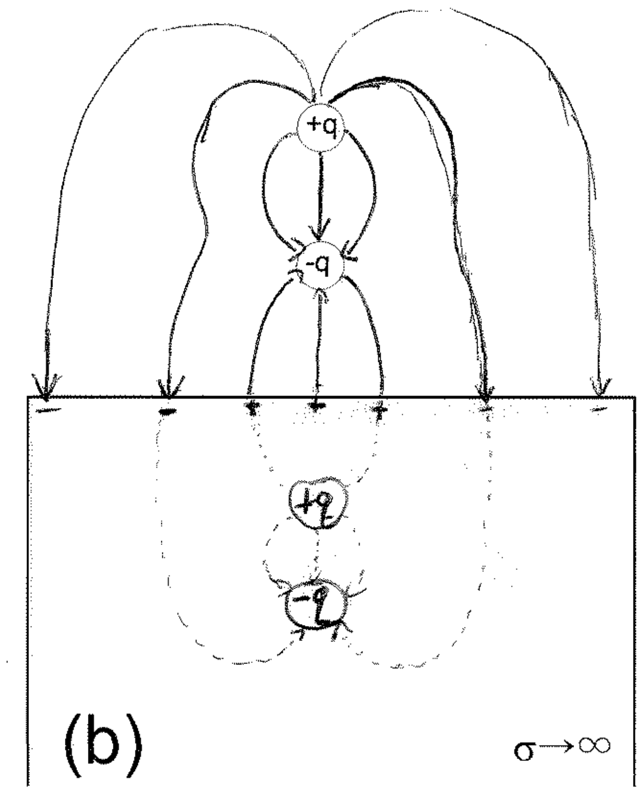
$L = \frac{\Phi}{I} = \frac{\mu\,\Delta z\, d}{W}$

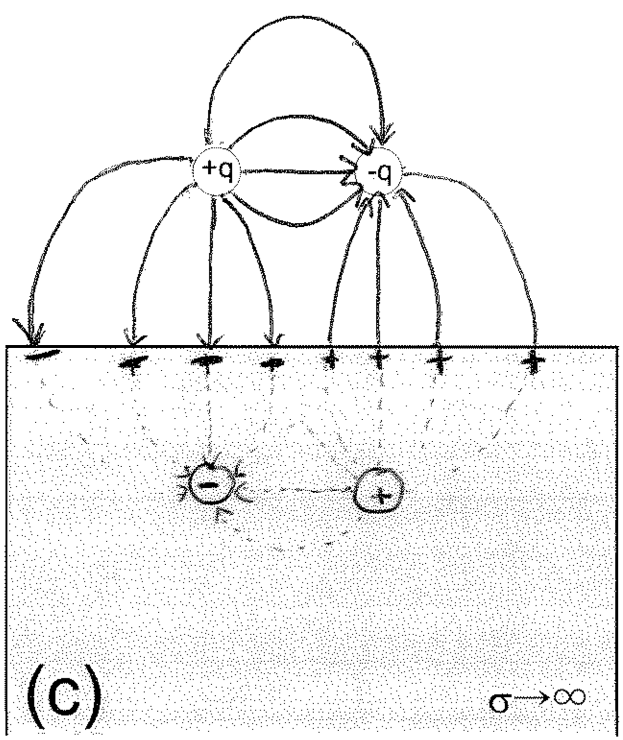
$\boxed{\frac{L}{\Delta z} = \frac{\mu d}{W}}$

4. Image Method (20 points)

Consider the following point charge configurations above a perfectly conducting ground plane (x-y plane). For each case, sketch the appropriate image charge configuration (location, amount, and polarity of charge). Also sketch the actual E-field lines that result from the presence of the ground plane, and the location and polarity of any surface charge. Be sure your sketch is neat and conveys the essential features of the field.




+q
-q
+q
-q
(b)
σ→∞


+q
-q
(c)
σ→∞

Maxwell's Equations:

$$\nabla \cdot \mathbf{D} = \rho_f$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Auxillary Fields:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

In linear media: $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ $\quad \mathbf{D} = \varepsilon \mathbf{E}$ $\quad \mathbf{M} = \chi_m \mathbf{H}$ $\quad \mathbf{B} = \mu \mathbf{H}$

Electrostatic Potential: $\mathbf{E} = -\nabla V$ Vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$

Electrodynamic Potential: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_V (\nabla \cdot \mathbf{A})\, dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$

Stokes's Theorem: $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{l}$

Electric energy density: $W_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$ or $W_e = \frac{1}{2} \varepsilon E^2$ (in linear media)

Magnetic energy density: $W_m = \frac{1}{2} \mathbf{B} \cdot \mathbf{H}$ or $W_m = \frac{1}{2} \mu H^2$ (in linear media)

Poynting Vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Time averaged Poynting vector: $\mathbf{S}_{av} = \frac{1}{2} \mathrm{Re}\left\{ \tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* \right\}$

Capacitance: $C = \frac{Q}{V}$

Inductance: $L = \frac{\Lambda}{I} = N \frac{\Phi}{I}$

Boundary conditions: $E_{t,2} - E_{t,1} = 0$ $\quad H_{t,1} - H_{t,2} = J_s$ $\quad D_{n,2} - D_{n,1} = \rho_s$ $\quad B_{n,2} - B_{n,1} = 0$

Bound charge: $\rho_{b,v} = -\nabla \cdot \mathbf{P}$ $\quad \rho_{b,s} = \mathbf{P} \cdot \hat{\mathbf{n}}$

Bound current: $\mathbf{J}_{b,v} = \nabla \times \mathbf{M}$ $\quad \mathbf{J}_{b,s} = \mathbf{M} \times \hat{\mathbf{n}}$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation, $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of $\mathbf{A}$, $\lvert\mathbf{A}\rvert =$	$+\sqrt{A_x^2 + A_y^2 + A_z^2}$	$+\sqrt{A_r^2 + A_\phi^2 + A_z^2}$	$+\sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product, $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $dv =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = +\sqrt{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = +\sqrt{x^2+y^2+z^2}$ $\theta = \tan^{-1}\left[+\sqrt{x^2+y^2}/z\right]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = +\sqrt{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

GRADIENT, DIVERGENCE, CURL, & LAPLACIAN OPERATORS

CARTESIAN (RECTANGULAR) COORDINATES (x, y, z)

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

CYLINDRICAL COORDINATES (r, ϕ, z)

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}}r & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} = \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{\mathbf{z}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

SPHERICAL COORDINATES (R, θ, ϕ)

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}}R & \hat{\boldsymbol{\phi}}R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & (R\sin\theta)A_\phi \end{vmatrix}$$

$$= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{R}\left[\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right]$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

SOME USEFUL VECTOR IDENTITIES

$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$ Scalar (or dot) product

$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin\theta_{AB}$ Vector (or cross) product, $\hat{\mathbf{n}}$ normal to plane containing $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(U + V) = \nabla U + \nabla V$$

$$\nabla(UV) = U\nabla V + V\nabla U$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$$

$$\nabla \cdot (U\mathbf{A}) = U\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla U$$

$$\nabla \times (U\mathbf{A}) = U\nabla \times \mathbf{A} + \nabla U \times \mathbf{A}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times \nabla V = 0$$

$$\nabla \cdot \nabla V = \nabla^2 V$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) d\mathcal{V} = \oint_S \mathbf{A} \cdot d\mathbf{s}$ Divergence theorem (S encloses $\mathcal{V}$)

$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\mathbf{l}$ Stokes's theorem (S bounded by C)