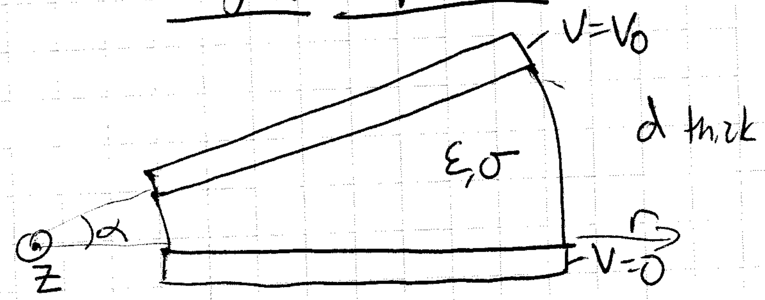
EE101 - Midterm Solutions - Winter 08

(1) Angled Capacitor



a) Potential Form $V = A\phi + B$ satisfies LaPlaces Eq inside cap.

$V(\phi=0)=0 \quad \Rightarrow \boxed{B=0}$

$V(\alpha)=V_0 \quad \Rightarrow A\alpha = V_0 \quad \Rightarrow \boxed{A = \frac{V_0}{\alpha}}$

$$\boxed{V(\phi) = \frac{V_0}{\alpha}\phi}$$

b) What is $\vec{E}$?

$$\vec{E} = -\nabla V = -\hat{\phi}\frac{1}{r}\frac{\partial V}{\partial \phi} = -\hat{\phi}\frac{V_0}{\alpha r}$$

c) What is surface charge density on upper & lower plates?

For a perfect conductor the boundary condition

$$\frac{\rho_s}{\varepsilon} = \vec{E}\cdot\hat{n}$$

at $\phi = 0 \quad \hat{n} = \hat{\phi} \quad \boxed{\rho_s = -\varepsilon\frac{V_0}{\alpha r}}$

at $\phi = \alpha \quad \hat{n} = -\hat{\phi} \quad \boxed{\rho_s = \varepsilon\frac{V_0}{\alpha r}}$

d) What is the capacitance?

Method 1: Find Q on each plate - use $C = \frac{Q}{V}$

$$Q = \int_0^d dz \int_{R_1}^{R_2} dr \left(\varepsilon\frac{V_0}{\alpha r}\right) = \frac{d\varepsilon V_0}{\alpha}\ln\left(\frac{R_2}{R_1}\right) \quad \text{on top plate}$$

$$Q = -\frac{d\varepsilon V_0}{\alpha}\ln\left(\frac{R_2}{R_1}\right) \quad \text{on bottom plate}$$

$$C = \frac{Q}{V} = \frac{d\varepsilon V_0}{\alpha V_0}\ln\frac{R_2}{R_1} \qquad \boxed{C = \frac{d\varepsilon}{\alpha}\ln\left(\frac{R_2}{R_1}\right)}$$

d) continued

Method 2: Calculate total electrostatic energy and compare with $W_e = \frac{1}{2}CV^2$

$$W_{e,total} = \frac{1}{2}CV^2 = \int_V \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\varepsilon \int_0^d dz \int_{R_1}^{R_2} dr \int_0^{\alpha} r\,d\phi \left(\frac{V_0}{\alpha r}\right)^2$$

$$W_{e,total} = \frac{\frac{1}{2}\varepsilon d \alpha V_0^2}{\alpha^2} \ln\left(\frac{R_2}{R_1}\right)$$

therefore $\boxed{C = \frac{\varepsilon d}{\alpha} \ln\left(\frac{R_2}{R_1}\right)}$

e) The resistance is inversely proportional to the capacitance

$$RC = \frac{\varepsilon}{\sigma}$$

$$R = \frac{\varepsilon}{\sigma}\frac{1}{C} = \frac{\varepsilon}{\sigma}\frac{\alpha}{\varepsilon d}\frac{1}{\ln\left(\frac{R_2}{R_1}\right)}$$

$$\boxed{R = \frac{\alpha}{\sigma d}\frac{1}{\ln(R_2/R_1)}}$$

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Which electrostatic field is impossible?

Electrostatic fields obey $\nabla \times \vec{E} = 0$

Case (a) $\nabla \times E = -2y\,\hat{x} - 3x\,\hat{y} - \hat{z} \neq 0$

Case (b) $\nabla \times E = 0$

Case (a) is impossible

③ Solenoid

a) Consider solenoid with $I = I_0 \cos \omega t$. What is **voltage** across terminals?

For an inductance from a loop with N coils $\quad V_{emf} = -N \frac{d\Phi}{dt} = -L \frac{dI}{dt}$

What is the self inductance of a solenoid?

Long solenoid approximation: Inside the B-field is uniform

$$\vec{B} = \frac{\mu_0 N I}{\ell} \hat{z}$$

Flux $\Phi = \frac{\pi b^2 \mu_0 N I}{\ell}$ $\qquad \boxed{L = \frac{\pi b^2 \mu_0 N^2}{\ell}}$

$$\boxed{V(t) = I_0 L \omega \sin \omega t}$$

b) Energy in magnetic field

$$\boxed{W_m = \frac{1}{2} L I^2 = \frac{\pi b^2 \mu_0 N^2}{2\ell} I_0^2 \cos^2 \omega t}$$

or

$$W_m = \frac{1}{2} \int_V \frac{B^2}{\mu_0} dV = \frac{1}{2} \frac{\pi b^2 \cancel{\ell}}{\cancel{\mu_0}} \frac{\mu_0^2 N^2}{\ell^2} I_0^2 \cos^2 \omega t$$

$$\boxed{W_m = \frac{\pi b^2 \mu_0 N^2}{2\ell} I_0^2 \cos^2 \omega t}$$

c) The changing flux from the solenoid will induce an emf in the wall of the copper pipe.

Assume $d \ll a$

~~The pipe~~

$$V_{emf} = \int_0^{2\pi} d\phi \, a E_\phi = -\frac{d\Phi_{21}}{dt}$$

due only to solenoid

Φ_{21} - flux through pipe due to solenoid

$$V_{emf} = -(\pi a^2) \frac{\mu_0 N}{l} \frac{dI}{dt}$$

$$\boxed{V_{emf} = \frac{\pi a^2 \mu_0 N \omega}{l} I_0 \sin(\omega t)}$$

This was all that is required for exam.

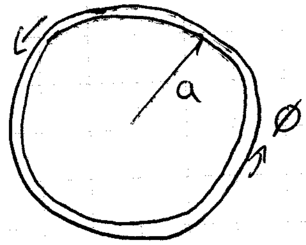
In reality, there will be a contribution from the flux generated by the pipe as well.

See next page for full solution (not required)

Full soln (not required)

c) The induced emf in the copper pipe will be proportional to the changing flux, both due to the solenoid, and the self contribution from the field from the copper pipe.

$$V_{emf} = \oint_0^{2\pi} d\phi\, a\, E_\phi = -\frac{d\Phi_2}{dt}$$



$$2\pi a\, E_\phi = -\underbrace{\frac{d\Phi_{21}}{dt}}_{\text{flux in pipe due to solenoid}} - \underbrace{\frac{d\Phi_{22}}{dt}}_{\text{self induced flux in pipe}}$$

$A_{pipe} = \pi a^2$

$A_{sol} = \pi b^2$

$$= -A_{pipe}\frac{\mu_0 N}{l}\frac{dI_1}{dt} - A_{pipe}\,\mu_0 \frac{dJ_\phi}{dt} d$$

Assume $J_\phi = \sigma E_\phi$

Assume Harmonic time dependence $e^{j\omega t}$

$$\frac{2\pi a}{\sigma} J_\phi = -A_{pipe}\frac{\mu_0 N}{l} j\omega I - A_{pipe}\,\mu_0\, j\omega J_\phi\, d$$

$$J_\phi\left(\frac{2\pi a}{\sigma} + j\omega A_{pipe}\mu_0 d\right) = -j\omega\frac{A_{pipe}\mu_0 N}{l} I$$

$$J_\phi = \frac{-j\omega A_{pipe}\mu_0 N I}{l\left(\frac{2\pi a}{\sigma} + j\omega A_{pipe}\mu_0 d\right)}$$

$$J_\phi = -j\omega\frac{A_{pipe}\mu_0 N I\sigma / l\, 2\pi a}{1 + j\omega\frac{d\, A_{pipe}\mu_0\sigma}{2\pi a}}$$

when σ is large

$$J_\phi = -\frac{NI}{l d} \Rightarrow \vec{J} = -\frac{NI}{ld}\ [illegible]$$

$2\pi a E_\phi = -$

$$J_\phi = \left[- \frac{\omega^2 a^2 \mu_0^2 N \; \sigma^2/4\ell}{1+\left(\frac{d a \mu_0 \sigma}{2}\right)^2} - j\frac{\omega a \mu_0 N \; \sigma/2\ell}{1+\left(\frac{d a \mu_0 \sigma}{2}\right)^2} \right] I$$

For a "good conductor" $\sigma \gg \frac{2}{d a \mu_0 \omega}$

The inductive response of the pipe dominates.

$J_\phi = -\frac{NI}{\ell d}$ which results in a field $B = -\frac{\mu_0 N I}{\ell}$ for $r<a$

This field is equal & opposite to that originally created by the solenoid $\Rightarrow$ this results in reduced flux $\rightarrow$ the Cu pipe screens the inside B-field.

For a "poor conductor" $\sigma \ll \frac{2}{d a \mu_0 \omega}$

The resistive response of the pipe dominates.

$J_\phi = \left(-j\omega \, a \mu_0 N \sigma / 2\ell\right) I$ $\quad -\frac{\pi}{2}$ out of phase with driving current in the solenoid

Thus $E \cdot J^*$ is real in pipe and the power is dissipated via Joule heating, not given back to the field.

d) The field created by the pipe opposed the solenoid field

$$B_{inside} = \frac{\mu_0 N I_0 \sin\omega t}{l} - \frac{\mu_0 N I_0 \sin\omega t}{l} = 0 \quad r<a$$

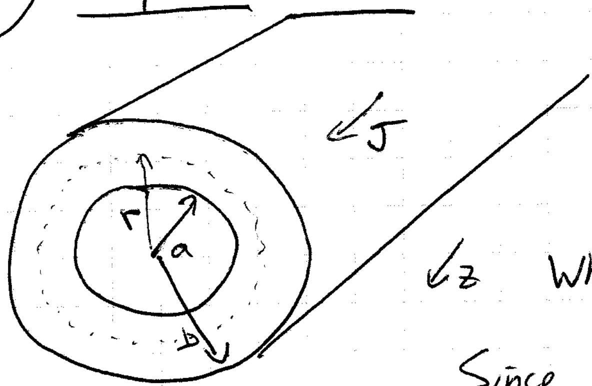
for $\sigma \to \infty$

$$B_{outside} = \frac{\mu_0 N I_0 \sin\omega t}{l} \qquad a<r<b$$

This results in less flux passing through solenoid.

The apparent self inductance will appear less, since the pipe has effectively screened the field inside of the pipe.

4 Ampere's Law



What is the H-field for $a<r<b$?

Since $\vec{J}=J\hat{z}$, the cylindrical symmetry tells us that $\vec{H}=H_\phi\hat{\phi}$

Choose circle for Ampere's Law

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} = I_{\text{total inside circle}}$$

$$d\vec{l} = r\,d\phi\,\hat{\phi}$$

$$\int_0^{2\pi} r\,d\phi\, H_\phi = \int_0^{2\pi} d\phi \int_a^r dr\, r\, J$$

$$2\pi r H_\phi = 2\pi \times \frac{1}{2}(r^2-a^2)J$$

$$\boxed{\vec{H} = \frac{J(r^2-a^2)}{2r}\hat{\phi}}$$