EE101 2007 Winter Midterm

Name: ______________________________

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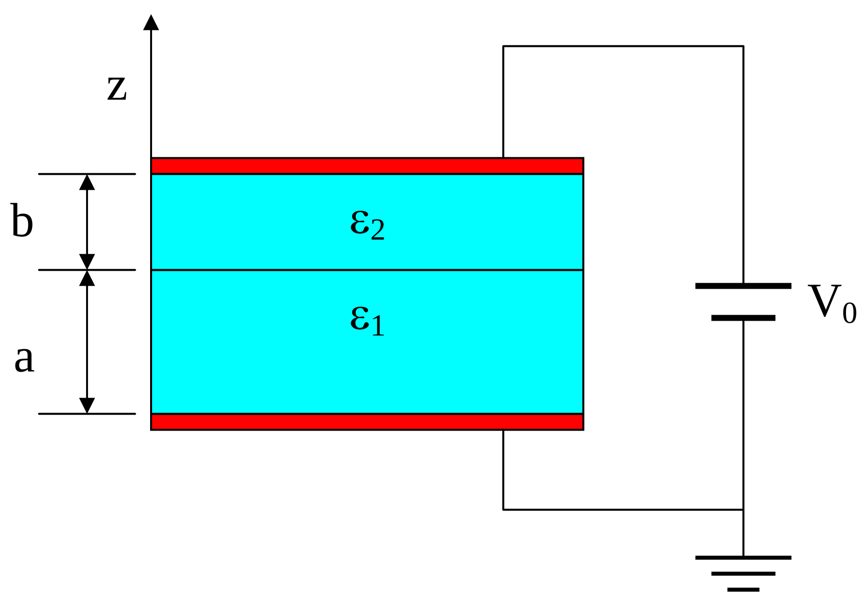
#1	
#2	
#3	
#4	
#5	
total	

Problem 1 (30 points)

The capacitor shown in figure consists of two parallel dielectric layers (ε_1 and ε_2). The bottom conductor plate is grounded and top plate is connected with voltage source V_0. Neglect the fringing field (field does not depends on variable x, y). The area of plate is A. Please find:

a). (7 oints) Electrical field (vector) distribution **E**(z) for the 0<=z<=a+b.
b). (7 points) Potential distribution V(z).
c). (5 point) The charge density and total charge Q on the top conductor plate.
d). (5 points) The capacitor capacitance.
e). (3 points) The electrostatic energy density.

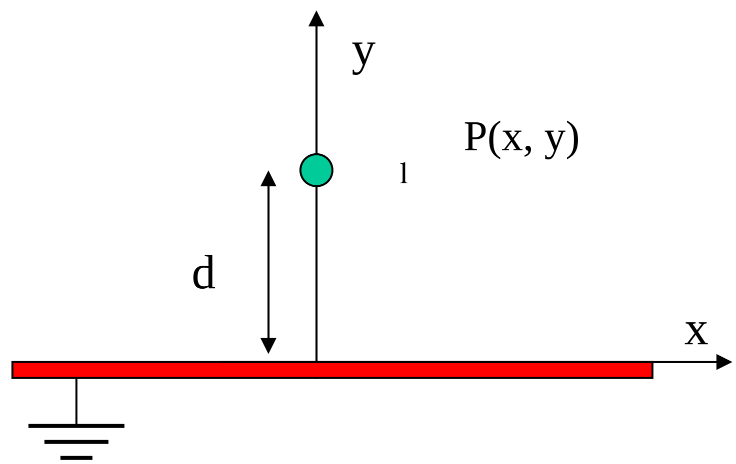
f). (3 points) Total electrostatic energy and prove it is equal to $V_0Q/2$.



Problem 2 (20 point)

A line charge ρ_l is place over infinite ground plane as shown in figure. Please find:

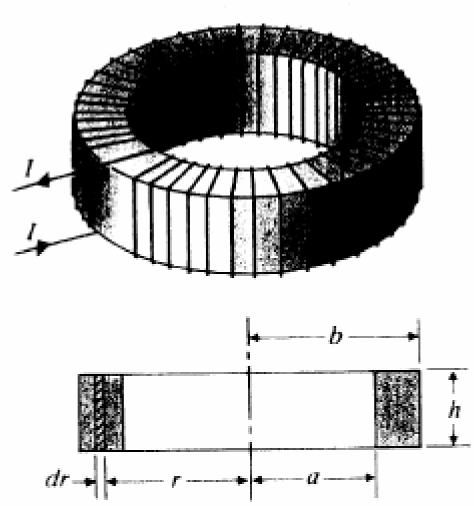
a). (15 point) Electric field intensity distribution $\mathbf{E}(x, y)$
b). (5 point) The induced charge density on the top surface of ground plane.



Problem 3 (20 points)

Assume that N turns of wire are **tightly** wound on toroidal frame of a rectangular cross section with dimension as shown in the Fig 3. Assume the permeability of the medium to be μ_o.

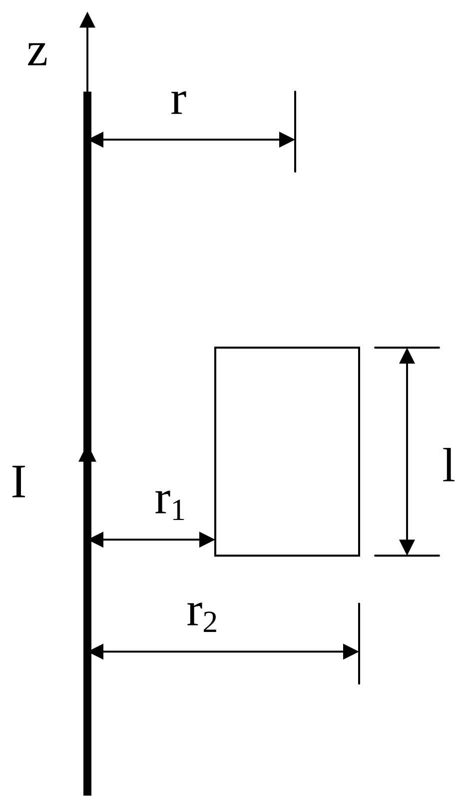
1. (10 points) find the magnetic field **H** and magnetic flux density B inside the toroidal. ($a<r<b$)
2. (10 points) find the self-inductance of the toroidal



Problem 4 (25 Point)

The rectangular loop of **N** turn is coplanar with long straight wire carrying the current I. Please find:

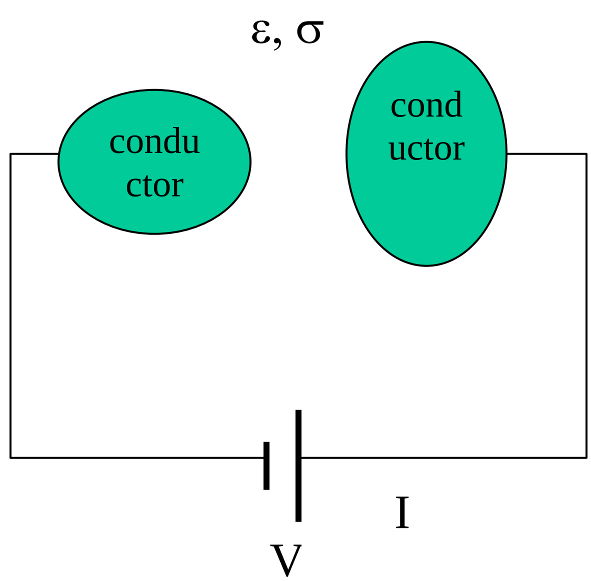
a). (5) The magnetic flux density **B**(r) due to straight wire (**need derivation**)
b). (10) The magnetic flux though the loop
c). (5) The magnetic flux linkage
d). (5) The mutual inductance L_{12}



Problem 5 (5 Points)

Prove the conductance G and capacitance of structure shown in the figure satisfy following relationship

$$G/C=\sigma/\varepsilon$$



EE101 2007 Winter Midterm Solution

a) Poisson Eq

$$\nabla^2 V_1 = 0 \qquad 0 \le x \le a$$

$$\nabla^2 V_2 = 0 \qquad a \le x \le a+b$$

boundary condition @ $z = a$

$$V_1(z)\big|_{z=a} = V_2(z)\big|_{z=a}$$

$$\varepsilon_1 \frac{dV_1(z)}{dz}\bigg|_{z=a} = \varepsilon_2 \frac{dV_2(z)}{dz}\bigg|_{z=a}$$

$z=0$ $V_1(z)=0$

$z=a+b$ $V_2(z)=0$

$$\nabla^2 V = 0 \quad \Rightarrow \quad \frac{d^2V}{dz^2} = 0$$

solution $\quad V_1(z) = C_{11} z + C_{12}$

$\qquad\qquad V_2(z) = C_{21} z + C_{22}$

Apply boundary condition

$z=0 \qquad V_1(z)\big|_{z=0} = 0$

$$C_{11} z + C_{12} = C_{12} = 0 \qquad (0)$$

$z = a+b \qquad V_2(z)\big|_{z=a+b} = V_0$

$$C_{21}(a+b) + C_{22} = V_0 \qquad (1)$$

$z = a \qquad V_1(z)\big|_{z=a} = V_2(z)\big|_{z=a}$

$$C_{11} a + C_{12} = C_{21} a + C_{22} \qquad (2)$$

$$\varepsilon_1 \frac{dV_1(z)}{dz}\bigg|_{z=a} = \varepsilon_2 \frac{dV_2(z)}{dz}\bigg|_{z=b}$$

$$\varepsilon_1 C_{11} = \varepsilon_2 C_{21} \qquad (2)$$

$(0) \rightarrow \quad C_{12} = 0$

$(2) \rightarrow \quad C_{21} = \frac{\varepsilon_1}{\varepsilon_2} C_{11}$

$(1) \rightarrow \quad C_{22} = V_0 - C_{21}(a+b) = V_0 - \frac{\varepsilon_1}{\varepsilon_2}(a+b) C_{11}$

(2)

$$C_{11} a + 0 = \frac{\varepsilon_1}{\varepsilon_2} C_{11} a + V_0 - \frac{\varepsilon_1}{\varepsilon_2}(a+b) C_{11}$$

$$= V - \frac{\varepsilon_1}{\varepsilon_2} C_{11} b$$

$$C_{11} = \frac{\varepsilon_2 V_0}{a\varepsilon_2 + \varepsilon_1 b}$$

$$C_{21} = \frac{\varepsilon_1 V_0}{a\varepsilon_2 + \varepsilon_1 b}$$

$$C_{22} = \frac{a V_0 (\varepsilon_2 - \varepsilon_1)}{a\varepsilon_2 + \varepsilon_1 b}$$

potential

$$\vec{E} = -\nabla V$$

$$\begin{cases} V_1(z) = C_{11} z + C_{12} \\ V_2(z) = C_{21} z + C_{22} \\ E_1(z) = -C_{11} \\ E_2(z) = -C_{21} \end{cases}$$

c) On the top plate

$$\rho_s = \vec{D} \cdot \hat{n} =$$

at $z = a+b$ $\quad \rho_{s2} = -\varepsilon_2 E_2 \hat{z} \cdot (-\hat{z}) = \varepsilon_2 E_2 = \dfrac{\varepsilon_1 \varepsilon_2 V_0}{\varepsilon_2 a + \varepsilon_1 b}$

Total charge

$$Q = \rho_s \cdot \Delta S = \rho_s A = \frac{\varepsilon_1 \varepsilon_2 V_0 A}{\varepsilon_2 a + \varepsilon_1 b}$$

d) Capacitor

$$C = \frac{Q}{V_0} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 a + \varepsilon_1 b}$$

e) electricostatic energy density

$$w_e = \frac{1}{2} \varepsilon E^2$$

$0 \le z \le a$ $\quad w_{e1} = \frac{1}{2} \varepsilon_1 E_1^2 = \frac{1}{2} \varepsilon_1 \left(\dfrac{\varepsilon_2 V_0}{a \varepsilon_2 + \varepsilon_1 b} \right)^2$

$a \le z \le a+b$ $\quad w_{e2} = \frac{1}{2} \varepsilon_2 E_2^2 = \frac{1}{2} \varepsilon_2 \left(\dfrac{\varepsilon_1 V_0}{a \varepsilon_2 + \varepsilon_1 b} \right)^2$

f) total electrostatic energy

$$W_1 = w_{e1} \cdot A d_1 = \frac{1}{2} \varepsilon_1 \left(\frac{\varepsilon_2 V_0}{a \varepsilon_2 + \varepsilon_1 b} \right)^2 A a.$$

$$W_2 = w_{e2} \cdot A d_1 = \frac{1}{2} \varepsilon_2 \left(\frac{\varepsilon_1 V_0}{a \varepsilon_2 + \varepsilon_1 b} \right)^2 A b$$

$$W_e = W_1 + W_2$$

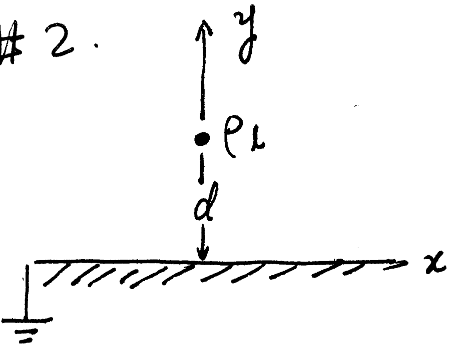
$$= \frac{1}{2}\varepsilon_1 \left(\frac{\varepsilon_2 V_0}{a\varepsilon_2 + \varepsilon_1 b}\right)^2 A\,a + \frac{1}{2}\varepsilon_2 \left(\frac{\varepsilon_1 V_0}{a\varepsilon_2 + \varepsilon_1 b}\right)^2 A\,b$$

$$= \frac{1}{2} V_0 \, A \, \frac{\varepsilon_1 \varepsilon_2 V_0}{(a\varepsilon_2 + \varepsilon_1 b)^2} (a\varepsilon_2 + \varepsilon_1 b)$$

$$= \frac{1}{2} V_0 \, A \, \frac{\varepsilon_1 \varepsilon_2 V_0}{(a\varepsilon_2 + \varepsilon_1 b)}$$

$$= \frac{1}{2} V_0 \, Q$$

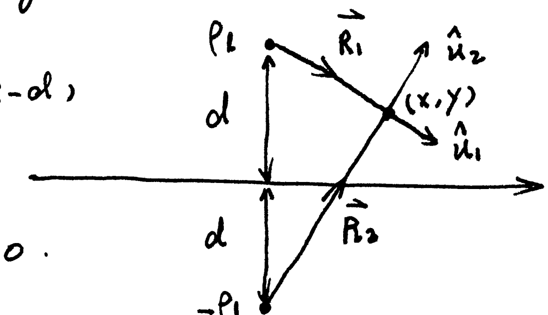
\# 2.



Method of Image

$\Longleftrightarrow$

$$\begin{cases} \nabla^2 V = -\frac{1}{\varepsilon_0}\delta(y-d) \\ V = 0 \big|_{y=0} \\ \text{in } \cancel{\text{I, II}}\ y \geq 0. \end{cases}$$



① $\vec{E}$: Due To ρ_l :

$$\vec{E}_1 = \hat{u}_1 \frac{\rho_l}{2\pi R_1 \varepsilon_0}$$

$$= \frac{\hat{x}x + \hat{y}(y-d)}{\sqrt{x^2+(y-d)^2}} \cdot \frac{\rho_l}{2\pi\varepsilon_0\sqrt{x^2+(y-d)^2}}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0} \cdot \frac{\hat{x}x + \hat{y}(y-d)}{x^2+(y-d)^2}$$

Due to $-\rho_l$: $\vec{E}_2 = \hat{u}_2 \cdot \frac{-\rho_l}{2\pi\varepsilon_0 R_2} = -\frac{\rho_l}{2\pi\varepsilon_0} \cdot \frac{\hat{x}x + \hat{y}(y+d)}{x^2+(y+d)^2}$

$$E = \vec{E}_1 + \vec{E}_2$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\left[\hat{x}\left(\frac{x}{x^2+(y-d)^2} - \frac{x}{x^2+(y+d)^2}\right) + \hat{y}\left(\frac{y-d}{x^2+(y-d)^2} - \frac{y+d}{x^2+(y+d)^2}\right)\right]$$

② $\rho_s = \varepsilon_0 E_y \big|_{y=0}$

$$= \frac{\cancel{\varepsilon_0}\rho_l}{2\pi}\left[\frac{y-d}{x^2+(y-d)^2} - \frac{y+d}{x^2+(y+d)^2}\right]\Bigg|_{y=0}$$

$$= \frac{-\rho_l d}{\pi(x^2+d^2)}$$

#3

a) Because symmetry $\vec{B}$ only has $\hat{\phi}$ component, B_ϕ which is only function of variable r

$$\vec{B} = \hat{\phi} B_\phi(r)$$

choose circular integral contour

$$\vec{dl} = \hat{\phi} r d\phi$$

Ampere's Law

$$\oint \vec{B} \cdot \vec{dl} = \mu I$$

$$= \int_0^{2\pi} B_\phi(r) \hat{\phi} \cdot \hat{\phi} r d\varphi = 2\pi r B_\varphi(r) = NI$$

$$\vec{B} = \hat{\phi} \frac{\mu_0 N I}{2\pi r} \hat{\phi}$$

b) $$\Phi = \int \vec{B} \cdot \vec{ds} = \int_s \hat{\phi} \frac{\mu_0 N I}{2\pi r} \hat{\phi} h dr$$

$$= \frac{\mu_0 N I}{2\pi} \ln \frac{b}{a}$$

$$\Lambda = N\phi$$

$$L = \frac{\Lambda}{I} = \frac{\mu_0 N^2 I}{2\pi} \ln\left(\frac{b}{a}\right) (h)$$

#4 ① Applying Ampere's law

$$\oint \vec{H} \cdot d\vec{l} = I \Rightarrow \vec{H} = \hat{\phi} \frac{I}{2\pi r}$$

$$\vec{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r}.$$

② $\Phi_{12} = \int_s \vec{B} \cdot d\vec{s}$

$$= \int_{r_2}^{r_1} \int_{z_0}^{z_0+l} \frac{\mu_0 I}{2\pi r} \, dr \, dz \quad (\text{select } \hat{\phi} \text{ as } \hat{n} \text{ of the surface})$$

$$= \frac{\mu_0 I l}{2\pi} \ln\left(\frac{r_2}{r_1}\right).$$

③ $\Lambda_{12} = N \Phi_{12} = \dfrac{\mu_0 I l N}{2\pi} \ln\left(\dfrac{r_2}{r_1}\right)$

$$L_{12} = \frac{\Lambda_{12}}{I} = \frac{N \mu_0 l}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

#5

$$\vec{j} = \sigma \vec{E}$$

Total current

$$I = \iint_{S_1} \vec{j} \cdot \vec{ds} = \sigma \iint_{S_1} \vec{E} \cdot \vec{ds}$$

Total charge on conductor 1

$$Q = \iint \rho_s \cdot ds = \iint \vec{D} \cdot \vec{ds} = \varepsilon \iint_{S_1} \vec{E} \cdot \vec{ds}$$

Voltage between conductor 1 & 2

$$V = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$$

$$G = \frac{I}{V} = \frac{\sigma \iint \vec{E} \cdot \vec{ds}}{-\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \iint \vec{E} \cdot \vec{ds}}{-\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}}$$

$$\boxed{\frac{G}{C} = \frac{\sigma}{\varepsilon}}$$