20'

Problem 1.

3' a) Find the dimensions of a hollow rectangular metallic waveguide for single mode operation within 1-1.5 GHz.

3' b) Determine the frequency range over which the waveguide supports dual modes.

5' c) Determine the waveguide guided modes at 4.5 GHz.

6' d) Find the group velocity, phase velocity, wave impedance, and wavelength of a 1.25 GHz wave propagating along the waveguide.

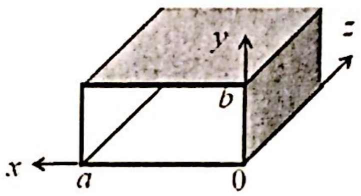
3' e) If the metallic waveguide in part (a) is filled with glass ($\varepsilon_r = 2.25$), find the frequency range for waveguide single mode operation.

a) First cutoff: $f_{01} = \frac{c}{2b} = 1\text{ GHz} \Rightarrow b = 0.15\text{m}$

Second cutoff: $f_{10} = \frac{c}{2a} = 1.5\text{GHz} \Rightarrow a = 0.1\text{m}$

(OK if a, b switch with each other, then $f_{c1} = f_{10}$, $f_{c2} = f_{01}$)

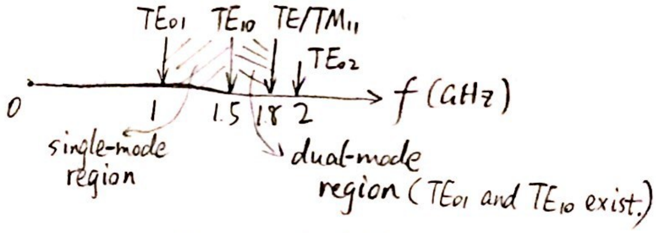
(If consider 1~1.5GHz is a sub-band of the single-mode operation band, then $a \le 0.1\text{m}$, $b \ge 0.15\text{m}$.)



(a<b)

b) $f_{11} = \frac{c}{2}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2} = 1.8\text{GHz}$

Dual-mode band: 1.5~1.8GHz



c) $f_{mn} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} < 4.5\text{GHz}$

$\Rightarrow \left(\frac{m}{0.1}\right)^2 + \left(\frac{n}{0.15}\right)^2 < 900$. The solutions are $\begin{cases} m=0,\ n=1,2,3,4 \\ m=1,\ n=0,1,2,3,4 \\ m=2,\ n=0,1,2,3 \end{cases}$

So the supported modes are $TE_{01\sim04}$, TE_{10}, $TE/TM_{11\sim14}$, TE_{20}, $TE/TM_{21\sim23}$

(TE_{30} has a cutoff frequency of exactly 4.5GHz. Strictly speaking, it won't be guided.)

(13 total cutoff frequencies, 20 modes can be guided at 4.5GHz, including 13 TE modes and 7 TM modes.)

d) When f=1.25GHz, only TE_{01} is guided as it is in the single-mode band.

$v_g = c\sqrt{1-(f_{01}/f)^2} = 3\times10^8\sqrt{1-(1/1.25)^2} = 1.8\times10^8\text{ m/s}$.

$v_p = \frac{c}{\sqrt{1-(f_{01}/f)^2}}$ or $v_p = \frac{c^2}{v_g} = 5\times10^8\text{ m/s}$.

$Z_{TE} = \eta/\sqrt{1-(f_{01}/f)^2} = 120\pi/\sqrt{1-(1/1.25)^2} = 200\pi = 628\ \Omega$.

$\lambda = \frac{2\pi}{\beta}$, where $\beta = \frac{\omega}{c}\sqrt{1-(f_{01}/f)^2} = \frac{2\pi\times1.25\times10^9}{3\times10^8}\sqrt{1-(1/1.25)^2} \Rightarrow \lambda = 0.4\text{m}$

or $\lambda = \frac{v_p}{f} = \frac{5\times10^8}{1.25\times10^9} = 0.4\text{m}$.

e) The single-mode band becomes $\frac{1}{\sqrt{2.25}} \sim \frac{1.5}{\sqrt{2.25}} = 0.67 \sim 1\text{GHz}$.

Generated by CamScanner

20'

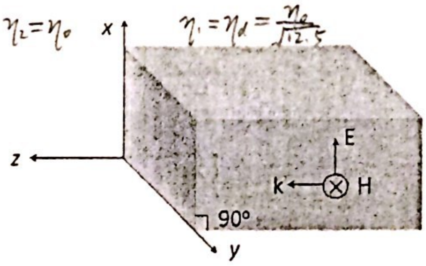
Problem 2.

An x-polarized optical beam is propagating in *z* direction inside a semi-infinite dielectric slab (ε_r = 12.5) placed in free space. The dielectric slab has rectangular sidewalls (as shown below) with dimensions much larger than the optical beam size.

4' a) If the optical beam is normally incident on one of the slab sidewalls in the xy-plane, determine the propagation direction and portion of the optical power that is transmitted from the slab to air.

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\eta_0 - \frac{\eta_0}{\sqrt{12.5}}}{\eta_0 + \frac{\eta_0}{\sqrt{12.5}}} = \frac{\sqrt{12.5}-1}{\sqrt{12.5}+1} = 0.56$$

$\eta_2 = \eta_0$ $\quad \eta_1 = \eta_d = \frac{\eta_0}{\sqrt{12.5}}$



Propagation direction $\hat{k}_t = \hat{z}$

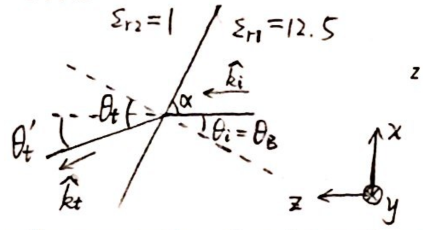
$$\frac{P^t}{P^i} = T = 1 - |\Gamma|^2 = 0.69$$

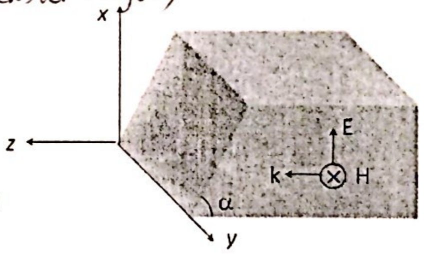
$$\text{or } T = |\tau|^2 \cdot \frac{\eta_1}{\eta_2} = |1+\Gamma|^2 \cdot \frac{1}{\sqrt{12.5}} = 0.69$$

4' b) If there is a possibility of polishing the slab sidewalls, find the slab sidewall angel, α, at which maximum optical power is transmitted from the slab to air.

$\Gamma_{\parallel} = 0 \Rightarrow \theta_B = \tan^{-1}\left(\sqrt{\frac{1}{12.5}}\right) = 15.8°$. (Brewster Angle)

$\alpha = 90° - \theta_B = 74.2°$





4' c) For part (b), determine the propagation direction and portion of the optical power that is transmitted from the slab to air. $\frac{\sin\theta_t}{\sin\theta_B} = \sqrt{12.5} \Rightarrow \theta_t = 73.74° \Rightarrow \theta_t' = \theta_t - \theta_B = 58°$.

$\hat{k}_t = -\hat{x}\sin\theta_t' + \hat{z}\cos\theta_t' = -\hat{x}\,0.85 + \hat{z}\,0.53$. $\quad \Gamma_{\parallel} = 0 \Rightarrow T_{\parallel} = 1$.

4' d) How can we maximize optical power transmission from the slab to air, if a y-polarized optical beam (instead of an x-polarized optical beam) is propagating in z direction.

$\Gamma_\perp$ keeps increasing with increasing θ_i. So maximum transmission occurs at $\theta_i = 0$ or $\alpha = 90°$ (normal incidence)

4' e) For the solution you offer for part (d), determine the propagation direction and portion of the optical power that is transmitted from the slab to air.

$\hat{k}_t = \hat{z}$. $\quad \Gamma_\perp = \Gamma$ in (a)

$= 0.56$.

$$T_\perp = 1 - |\Gamma_\perp|^2 = 0.69.$$

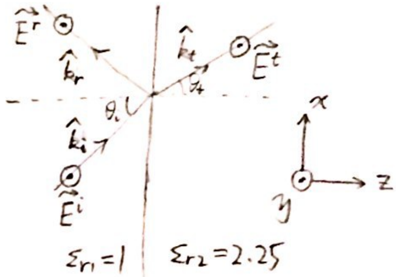
20′ **Problem 3.**

A 1GHz, y-polarized plane-wave is propagating in air along $\hat{k} = \frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{z}$ direction. The wave is incident on a semi-infinite dielectric medium with $\mu = \mu_0$ and $\varepsilon_r = 2.25$. The interface between the dielectric medium and air is the z = 0 plane.

5′+6′×2 a) Write the phasor domain and time domain equations for the incident and reflected wave in air and the transmitted wave into the dielectric medium.

3′ b) Determine the portion of the wave power that is transmitted into the dielectric medium.

a) $\vec{E}^i = \hat{y}E_0 e^{-j\vec{k}_i\cdot\vec{r}}$ $\quad (\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z)$



$k_i = \frac{\omega}{c} = \frac{2\pi\times10^9}{3\times10^8} = \frac{20\pi}{3}$, $\hat{k}_i = \frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{z}$ as given. $(\theta_i = 60°)$

So $\vec{k}_i = k_i\cdot\hat{k}_i = \frac{10\sqrt{3}\pi}{3}\hat{x} + \frac{10\pi}{3}\hat{z}$.

$\vec{E}^i = \hat{y}E_0 e^{-j(\frac{10\sqrt{3}\pi}{3}x + \frac{10\pi}{3}z)}$.

$\vec{E}^i = Re\{\vec{E}^i e^{j\omega t}\} = \hat{y}E_0\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x - \frac{10\pi}{3}z)$

$\eta_i = \eta_0 = 120\pi$ or 377Ω. $\quad \vec{H}^i = \frac{1}{\eta_i}\hat{k}_i\times\vec{E}^i = \frac{1}{377}(\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{z})\times\hat{y}E_0 e^{-j\vec{k}_i\cdot\vec{r}}$

$= (-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z})\frac{E_0}{377}e^{-j(\frac{10\sqrt{3}\pi}{3}x + \frac{10\pi}{3}z)}$

$\vec{H}^i = (-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{z})\frac{E_0}{377}\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x - \frac{10\pi}{3}z)$.

→ Reflected wave:

$k_r = k_i = \frac{20\pi}{3}$, $\hat{k}_r = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z}$. $\Gamma_\perp = \frac{\cos60° - \sqrt{2.25 - \sin^2 60°}}{\cos60° + \sqrt{2.25 - \sin^2 60°}} = -0.42$.

$\vec{E}^r = \hat{y}\Gamma_\perp E_0 e^{-j\vec{k}_r\cdot\vec{r}} = -\hat{y}0.42E_0 e^{-j(\frac{10\sqrt{3}\pi}{3}x - \frac{10\pi}{3}z)}$

$\vec{E}^r = -\hat{y}0.42E_0\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x + \frac{10\pi}{3}z)$.

$\eta_r = \eta_i = \eta_0 = 377$. $\quad \vec{H}^r = \frac{1}{\eta_r}\hat{k}_r\times\vec{E}^r = \frac{1}{377}(\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{z})\times(-\hat{y})0.42E_0 e^{-j\vec{k}_r\cdot\vec{r}}$

$= (-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z})0.42\frac{E_0}{377}e^{-j(\frac{10\sqrt{3}\pi}{3}x - \frac{10\pi}{3}z)}$

$\vec{H}^r = (-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{z})0.42\frac{E_0}{377}\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x - \frac{10\pi}{3}z)$.

b) $\frac{P^t}{P^i} = T_\perp = 1 - |\Gamma_\perp|^2$
$= 0.82$
or $T_\perp = |\tau_\perp|^2\frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}$
$= (0.58)^2\frac{\sqrt{2.25}\cos(35.3°)}{\cos60°}$
$= 0.82$.

→ Transmitted wave:

$k_t = k_i\sqrt{2.25} = 10\pi$. $\quad \sin\theta_t = \frac{\sin\theta_i}{\sqrt{2.25}} = \frac{\sqrt{3}}{3} \Rightarrow \theta_t = 35.3°$

$\hat{k}_t = \hat{x}\sin\theta_t + \hat{z}\cos\theta_t = \frac{\sqrt{3}}{3}\hat{x} + \frac{\sqrt{6}}{3}\hat{z}$ $\quad \tau_\perp = 1 + \Gamma_\perp = 0.58$.

$\vec{E}^t = \hat{y}\tau_\perp E_0 e^{-j\vec{k}_t\cdot\vec{r}} = \hat{y}0.58E_0 e^{-j(\frac{10\sqrt{3}\pi}{3}x + \frac{10\sqrt{6}\pi}{3}z)}$

$\vec{E}^t = \hat{y}0.58E_0\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x - \frac{10\sqrt{6}\pi}{3}z)$

$\eta_t = \frac{\eta_0}{\sqrt{2.25}} = 80\pi$. $\quad \vec{H}^t = \frac{1}{\eta_t}\hat{k}_t\times\vec{E}^t = \frac{1}{80\pi}(\frac{\sqrt{3}}{3}\hat{x} + \frac{\sqrt{6}}{3}\hat{z})\times\hat{y}0.58E_0 e^{-j\vec{k}_t\cdot\vec{r}}$

$= (-\frac{\sqrt{6}}{3}\hat{x} + \frac{\sqrt{3}}{3}\hat{z})0.58\frac{E_0}{80\pi}e^{-j(\frac{10\sqrt{3}\pi}{3}x + \frac{10\sqrt{6}\pi}{3}z)}$

$\vec{H}^t = (-\frac{\sqrt{6}}{3}\hat{x} + \frac{\sqrt{3}}{3}\hat{z})0.58\frac{E_0}{80\pi}\cos(2\pi\times10^9 t - \frac{10\sqrt{3}\pi}{3}x - \frac{10\sqrt{6}\pi}{3}z)$.