CS 181 EXAM #3 **NAME** ______________________________

SPRING 2015 **UCLA ID** ______________________________

You have 90 minutes to complete this exam. You may assume without proof any statement proved in class.

1 Consider the following context-free grammar:

$$S \to SS \mid T$$
$$T \to aT \mid aTb \mid ab \mid a.$$

(1 pts) **a.** Describe the language generated by this grammar.

(1 pts) **b.** Prove that this grammar is ambiguous.

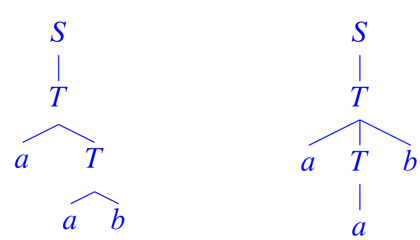
(2 pts) **c.** Give an equivalent unambiguous grammar.

Solution.

a. Nonempty strings of the form $a^{N_1}b^{n_1}a^{N_2}b^{n_2}\ldots a^{N_k}b^{n_k}$, where

$$\begin{gathered} N_1 \geqslant n_1, \\ N_2 \geqslant n_2, \\ \vdots \\ N_k \geqslant n_k. \end{gathered}$$

b. The string aab has at least two parse trees:



c. $S \to AS \mid TS \mid A \mid T$
$A \to aA \mid a$
$T \to aTb \mid ab.$

2 Give context-free grammars for the following languages over the binary alphabet:

(2 pts) **a.** nonempty even-length strings with the two middle symbols equal

(2 pts) **b.** strings with twice as many 0s as 1s

(3 pts) **c.** $\left\{0^n 1^m : n < m < \frac{2015}{2014} n\right\}$.

Solution:

a. $S \to \Sigma S \Sigma \mid 00 \mid 11$
$\Sigma \to 0 \mid 1$

b. $S \to 1S0S0S \mid 0S1S0S \mid 0S0S1S \mid \varepsilon$

c. $S \to 0S1 \mid 0T1$
$T \to 0^{2014} T 1^{2015} \mid 0^{2014} 1^{2015}$

3 True or false? Prove your answer.

(2 pts) **a.** If L is context-free, then the set of all substrings of strings in L is a context-free language.

(3 pts) **b.** If L is not context-free and F is finite, then $L \setminus F$ is not context-free.

Solution.

a. True. The set of all substrings of strings in L is prefix(suffix(L)), which is context-free whenever L is context-free (by the closure of context-free languages under prefix and suffix).

b. True. We will prove the contrapositive: if F is finite and $L \setminus F$ context-free, then L is context-free. For this, write

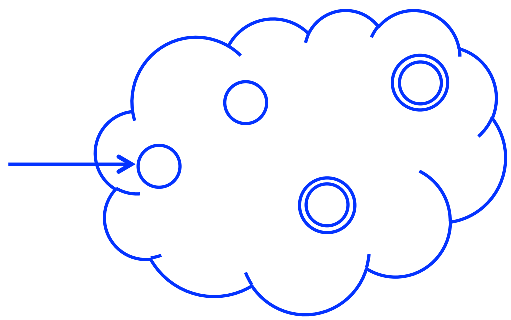
$$L = (L \setminus F) \cup (L \cap F).$$

For any finite F, the language $L \cap F$ is also finite, hence regular, hence context-free. We conclude that, with F finite and $L \setminus F$ context-free, L is the union of two context-free languages and is therefore itself context-free (by the closure of context-free languages under union).

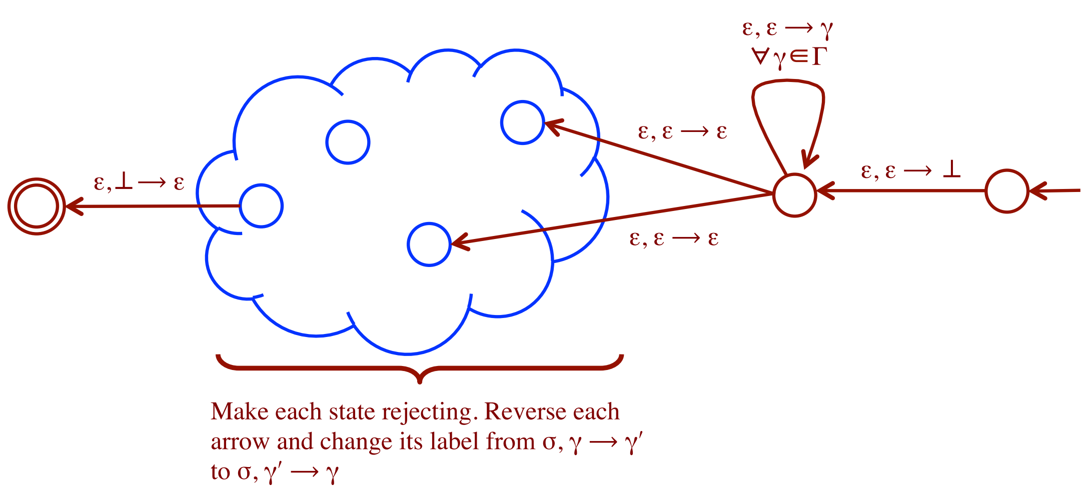
(3 pts) **4** Let L be a given context-free language. Explain how to obtain a PDA for reverse(L) from a PDA for L.

Your solution must not involve context-free grammars in any way. In particular, the following argument must not be used: convert the PDA to a grammar, reverse each rule, and convert back to a PDA.

Solution. The transformation is similar to that for DFAs and NFAs, except that one must now be careful not to forget about the stack. Suppose that, schematically, the original PDA $(Q, \Sigma, \Gamma, \delta, q_0, F)$ looks like this:



Fix a symbol, say $\perp$, that is not currently in Γ. To obtain a PDA for reverse(L), we make the structural changes shown in red:



The added loop plays a vital role in this construction. Its purpose is to "guess" the final contents of the stack for some accepting computation and to populate the stack accordingly.

5 For each of the following languages over the binary alphabet, determine whether it is context-free and prove your answer:

(2 pts) **a.** $\{wvw : w \in \{0,1\}^+, v \in \{0,1\}^*\}$

(2 pts) **b.** $\{0^n 1^m 0^k 1^{n+m} : n, m, k \geqslant 0\}$

(2 pts) **c.** palindromes with equally many 0s and 1s.

Solution. In all parts, L stands for the language is question.

a. Not context-free. Take an arbitrary integer $p \geqslant 1$ and consider the string $w = 0^p 1^p 0^p 1^p \in L$. Fix any decomposition $w = uvxyz$ for some strings u, v, x, y, z with $|v| + |y| \neq 0$ and $|vxy| \leqslant p$. There are two cases to examine: (i) if vxy is contained entirely within the first p symbols or entirely within the last p symbols, then $uv^2xy^2z \notin L$ (here, it is crucial that we pump *up* rather than *down*); (ii) if vxy overlaps with the middle $2p$ characters of w, then $uxz \notin L$. By the pumping lemma, L is not context-free.

b. Context-free, with grammar

$$\begin{aligned} S &\to 0S1 \mid T \\ T &\to 1T1 \mid U \\ U &\to 0U \mid \varepsilon. \end{aligned}$$

c. Not context-free. Take an arbitrary integer $p \geqslant 1$ and consider the string $1^p 0^{2p} 1^p \in L$. Fix any decomposition $1^p 0^{2p} 1^p = uvxyz$ for some strings u, v, x, y, z with $|v|+|y| \neq 0$ and $|vxy| \leqslant p$. There are two cases to examine: (i) if $v \in 0^*$ and $y \in 0^*$, then uv^2xy^2z contains more 0s than 1s and hence is not in L; (ii) if v or y contains a 1, then the length restriction $|vxy| \leqslant p$ implies that uxz contains unequal numbers of 1s on the left and on the right and therefore is not a palindrome: $uxz \notin L$. By the pumping lemma, L is not context-free.